Small positive values for supercritical branching processes in random environment

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Abstract

Branching Processes in Random Environment (BPREs) $(Z_n : n \geq 0)$ are the generalization of Galton-Watson processes where in each generation the reproduction law is picked randomly in an i.i.d. manner. In the supercritical case, the process survives with positive probability and then almost surely grows geometrically. This paper focuses on rare events when the process takes positive but small values for large times.

We describe the asymptotic behavior of $\mathbb{P}(1 \leq Z_n \leq k | Z_0 = i)$, $k, i \in \mathbb{N}$ as $n \to \infty$. More precisely, we characterize the exponential decrease of $\mathbb{P}(Z_n = k | Z_0 = i)$ using a spine representation due to Geiger. We then provide some bounds for this rate of decrease.

If the reproduction laws are linear fractional, this rate becomes more explicit and two regimes appear. Moreover, we show that these regimes affect the asymptotic behavior of the most recent common ancestor, when the population is conditioned to be small but positive for large times.

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1 Introduction

A branching process in random environment (BPRE) is a discrete time and discrete size population model going back to [33, 7]. In each generation, an offspring distribution is picked at random, independently from one generation to the other. We can think of a population of plants having a one-year life-cycle. In each year, the outer conditions vary in a random fashion. Given these conditions, all individuals reproduce independently according to the same mechanism. Thus, it satisfies both the Markov and branching properties.

Recently, the problems of rare events and large deviations have attracted attention [28, 9, 12, 29, 23, 10]. However, the problem of small positive values has not been treated except in the easiest case which assumes non-extinction, i.e. $\mathbb{P}(Z_1 \geq 1 | Z_0 = 1) \geq 1$ (see [9]). For Galton-Watson processes, the explicit equivalent of this probability is well-known (see e.g. [8][Chapter I, Section 11, Theorem

3]). In particular, denoting by f the probability generating function of the reproduction law, we have for k large enough,

$$\varrho := \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(1 \le Z_n \le k | Z_0 = 1) = f'(p_e), \text{ where } \quad p_e = \mathbb{P}(\exists n \in \mathbb{N} : Z_n = 0 \mid Z_0 = 1). \quad (1.1)$$

Moreover, the rate of decrease remains equal to ϱ if k_n decreases sub-exponentially. It means that as soon as $k_n/\exp(\delta n) \to 0$ as $n \to \infty$ for every $\delta > 0$, then $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(1 \le Z_n \le k_n \mid Z_0 = 1) = \varrho$. In this paper, we focus on the existence and characterization of ϱ in the random environment framework. It is organized as follows.

First, we give the classical notations and properties of BPRE. In the next section, we state our results. We prove the existence of ϱ and a characterization of its value via a spine construction, give a lower bound and an upper bound which have natural interpretations. Finally, we specify our results in the linear fractional case, where two regimes appear, which are also visible in the time of the most recent common ancestor (MRCA).

In the rest of the paper, the proofs of these results are presented. Section 3 deals with a tree construction due to Geiger, which is used in Section 4 to characterize ρ. In Section 5.2, we prove that $\rho > 0$ under suitable assumptions. In Section 5.3, we prove a lower bound for ϱ in terms of the rate function of the associated random walk. Finally, in Sections 6.1 and 6.2 the statements for the linear fractional case are proved using the general results obtained before, whereas in Section 7, we present some details on two examples.

For the formal definition of a branching process Z in random environment, let Q be a random variable taking values in Δ, the space of all probability measures on $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We denote by

$$m_q := \sum_{k\ge 0} k \; q(\{k\})$$

the mean number of offsprings of $q \in \Delta$. For simplicity of notation, we will shorten $q(\{\cdot\})$ to $q(\cdot)$ throughout this paper. An infinite sequence $\mathcal{E} = (Q_1, Q_2, \ldots)$ of independent, identically distributed (i.i.d.) copies of Q is called a random environment. Then the integer valued process $(Z_n : n \ge 0)$ is called a branching process in the random environment $\mathcal{E}$ if Z_0 is independent of $\mathcal{E}$ and it satisfies

$$\mathcal{L}\big(Z_n \mid \mathcal{E},\ Z_0, \ldots, Z_{n-1}\big) \;=\; Q_n^{*Z_{n-1}} \qquad \text{a.s.} \tag{1.2}$$

for every $n \ge 1$, where q^{*z} is the z-fold convolution of the measure q. We introduce the probability generating function (p.g.f) of Q_n, which is denoted by f_n and defined by

$$f_n(s) := \sum_{k=0}^{\infty} s^k Q_n(k), \qquad (s \in [0,1]).$$

In the whole paper, we denote indifferently the associated random environment by $\mathcal{E} = (f_1, f_2, ...)$ and $\mathcal{E} = (Q_1, Q_2, ...)$. The characterization (1.2) of the law of Z becomes

$$\mathbb{E}\big[s^{Z_n} | \mathcal{E},\ Z_0, \ldots, Z_{n-1}\big] = f_n(s)^{Z_{n-1}} \qquad \text{a.s.} \qquad (0 \le s \le 1, n \ge 1).$$

Many properties of Z are mainly determined by the **random walk associated with the environment** $(S_n \;:\; n \in \mathbb{N}_0)$ which is defined by

$$S_0 = 0, \qquad S_n - S_{n-1} = X_n \quad (n \ge 1),$$

where

$$X_n := \log m_{Q_n} = \log f_n'(1)$$

are i.i.d. copies of the logarithm of the mean number of offsprings $X := \log(m_Q) = \log(f'(1))$. Thus, one can check easily that

$$\mathbb{E}[Z_n | Q_1, \ldots, Q_n, Z_0 = 1] \quad = \quad e^{S_n} \quad \text{a.s.} \tag{1.3}$$

There is a well-known classification of BPRE [7], which we recall here in the case $\mathbb{E}[|X|] < \infty$. In the subcritical case ($\mathbb{E}[X] < 0$), the population becomes extinct at an exponential rate in almost every environment. Also in the critical case ($\mathbb{E}[X] = 0$), the process becomes extinct a.s. if we exclude the degenerated case when $\mathbb{P}(Z_1 = 1 | Z_0 = 1) = 1$. In the supercritical case ($\mathbb{E}[X] > 0$), the process survives with positive probability under quite general assumptions on the offspring distributions (see [33]). Then $\mathbb{E}[Z_1 \log^+(Z_1)/f'(1)] < \infty$ ensures that the martingale $e^{-S_n} Z_n$ has a positive finite limit on the non-extinction event:

$$\lim_{n\to\infty} e^{-S_n} Z_n = W \text{ a.s.}, \qquad \mathbb{P}(W > 0) = \mathbb{P}(\forall n \in \mathbb{N} : Z_n > 0 | Z_0 = 1) > 0.$$

The problem of small positive values is linked to the left tail of W and the existence of harmonic moments. In the Galton-Watson case, we refer to [6, 32, 17]. For BPRE, Hambly [22] gives the tail of W in 0, whereas Huang & Liu [23, 24] have obtained other various results in this direction.

2 Probability of staying bounded without extinction

Given the initial population size k, the associated probability is denoted by $\mathbb{P}_k(\cdot) := \mathbb{P}(\cdot | Z_0 = k)$. For convenience, we write $\mathbb{P}(\cdot)$ when the size of the initial population is taken equal to 1 or does not matter. Let $f_{i,n}$ be the probability generating function of Z_n started in generation $i \le n$:

$$f_{i,n} := f_{i+1} \circ f_{i+2} \circ \cdots \circ f_n, \quad f_{n,n} = Id \qquad \text{a.s.}$$

We will now specify the asymptotic behavior of $\mathbb{P}_i(Z_n = j)$ for $i, j \ge 1$, which may depend both on i and j. One can first observe that some integers j cannot be reached by Z starting from i owing to the support of the offspring distribution.
The first result below introduces the rate of decrease ϱ of $\mathbb{P}_i(Z_n = j)$ for $i, j \ge 1$ and gives a trajectorial interpretation of the associated rare event $\{Z_n = j\}$. The second one provides general conditions to ensure that $\varrho > 0$. It also gives an upper bound of ϱ, which may be reached, in terms of the rate function of the random walk S. This bound corresponds to the environmental stochasticity, which means that the rare event $\{Z_n = j\}$ is explained by rare environments. The next result yields an explicit expression of the rate ϱ in the case of linear fractional offspring distributions, where two supercritical regimes appear. The last corollary considers the most recent common ancestor, where a third regime appears which is located at the borderline of the phase transition.

Let us define

$$\mathcal{I} := \big\{ j \ge 1 \ : \ \mathbb{P}(Q(j) > 0, Q(0) > 0) > 0 \big\}$$

and introduce the set $Cl(\{z\})$ of integers that can be reached from $z \in \mathcal{I}$, i.e.

$$Cl(\{z\}) := \big\{k \geq 1 : \exists n \geq 0 \text{ with } \mathbb{P}_z(Z_n = k) > 0\big\}.$$

Analogously, we define $Cl(\mathcal{I})$ as the set of integers which can be reached from $\mathcal{I}$ by the process Z. More precisely,

$$Cl(\mathcal{I}) := \big\{k \geq 1 : \exists n \geq 0 \text{ and } j \in \mathcal{I} \text{ with } \mathbb{P}_j(Z_n = k) > 0\big\}.$$

We observe that $\mathcal{I} \subset Cl(\mathcal{I})$ and if $\mathbb{P}(Q(0) + Q(1) < 1) > 0$ and $\mathbb{P}(Q(0) > 0,\ Q(1) > 0) > 0$, then $Cl(\mathcal{I}) = \mathbb{N}$.

We are interested in the event $\{Z_n = j\}$ for large n. First, we recall that the case $\mathbb{P}(Z_1 = 0) = 0$ is easier and the rate of decrease of the probability is known [9]. Indeed, then Z is nondecreasing and for $k \geq j \in \mathbb{N}$ such that $\mathbb{P}_k(Z_l = j) > 0$ for some $l \geq 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_k(Z_n = j) = -k\varrho, \qquad \text{with } \varrho = -\log \mathbb{P}_1(Z_1 = 1).$$

We note that in the case $\mathbb{P}_1(Z_1 = 1) = \mathbb{P}(Z_1 = 0) = 0$, the branching process grows exponentially in almost every environment and the probability on the left-hand side is zero if n is large enough. Thus, let us now focus on the supercritical case with possible extinction, which ensures that $\mathcal{I}$ is not empty. The expression of ρ in the next theorem will be used to get the other forthcoming results.

Theorem 2.1. *We assume that $\mathbb{E}[X] > 0$ and $\mathbb{P}(Z_1 = 0) > 0$. Then the following limits exist and coincide for all $k, j \in Cl(\mathcal{I})$,*

$$\varrho := -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_k(Z_n = j) = -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\big[Q_n(z_0) f_{0,n}(0)^{z_0-1} \Pi_{i=1}^{n-1} f_i'\big(f_{i,n}(0)\big)\big]$$

where z_0 is the smallest element of $\mathcal{I}$. The common limit ϱ belongs to $[0, \infty)$.

The proof is given in Section 4 and results from Lemmas 4.1 and 4.2.
The right-hand side expression of ϱ correspond to the event $\{Z_n = j\}$ explained by a "spine structure". More precisely, one individual survives until generation n and gives birth to the j survivors in the very last generations, whereas the other subtrees become extinct (see forthcoming Lemma 3.2 for details). However, we are seeing in the linear fractional case (Corollary 2.3) that a multi-spine structure can also explain $\{Z_n = j\}$ in some regime. Thus the optimal strategy is nontrivial and will here only be discussed in the linear fractional case.
The proof of Theorem 2.1 is easy if we consider the limit of $\frac{1}{n} \log \mathbb{P}_1(Z_n = 1)$ as $n \to \infty$. In this case, a direct calculation of the first derivative of $f_{0,n}$ yields the claim. However, the proof for the general case is more involved. Here, we use probabilistic arguments, which rely on a spine decomposition of the conditioned branching tree via Geiger construction.

We also note that we need to focus on $i, j \in Cl(\mathcal{I})$. Indeed, $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1(Z_n = i)$ and $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1(Z_n = j)$ may both exist and be finite for $i \neq j$, but have different values. To see that, one can consider two environments q_1 and q_2 such that

$$\mathbb{P}(Q_1 = q_1) = 1 - \mathbb{P}(Q_1 = q_2) > 0; \quad q_1(1) = 1; \quad q_2(0) + q_2(2) = 1.$$

Moreover, the case $-\infty < \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_k(Z_n = k) < \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1(Z_n = k) < 0$ with $k > 1$ is also possible. These results are developed in the two examples given in Section 7 at the

end of this paper.

In the Galton-Watson case, f is constant and for every $i \geq 0$, $f_i = f$ a.s. Then $f_{i,n}(0) \to p_e$ as $n \to \infty$ and we recover the classical result (1.1).

The results and remarks above could lead to the conjecture $\varrho = -\log \mathbb{E}[f'(p(f))]$, where $p(f) = \inf\{s \in [0,1] : f(s) = s\}$. Roughly speaking, it would correspond to integrate the value obtained in the Galton-Watson case with respect to the environment. The two following results show that this is not true in general.

To prove that the probability of staying small but alive decays exponentially (i.e. $\rho > 0$) requires some assumptions. To avoid too much technicalities, we are assuming

Assumption 1. *There exists $\gamma > 0$ such that $Q(0) < 1 - \gamma$ a.s. and $\mathbb{E}[|X|] < \infty$.*

Similarly, to give an upper bound of ϱ in terms of the rate function of the random walk S, we require the following Assumption. The non-lattice condition is only required for a functional limit result which is taken from [2], whereas the truncated moment assumption is classically used for lower bounds of the survival probability of BPREs.

Assumption 2. *We assume that S is non-lattice, i.e. for every $r > 0$, $\mathbb{P}(X \in r\mathbb{Z}) < 1$. Moreover, there exist $\varepsilon > 0$ and $a \in \mathbb{N}$ such that for every $x > 0$,*

$$\mathbb{E}\big[(\log^+ \xi_Q(a))^{2+\varepsilon} | X > -x\big] < \infty,$$

where $\log^+ x := \log(\max(x, 1))$ and $\xi_q(a)$ is the truncated standardized second moment

$$\xi_q(a) := \sum_{y=a}^{\infty} y^2 q(y)/m_q^2, \quad a \in \mathbb{N}, q \in \Delta.$$

Proposition 2.2. *We assume that there exists $s > 0$ such that $\mathbb{E}[e^{-sX}] < \infty$.*
(i) If Assumption 1 is fulfilled, then $\rho > 0$.
(ii) If $\mathbb{P}(X \geq 0) = 1$ or Assumption 2 holds, then

$$\varrho \leq -\log \inf_{\lambda \geq 0} \mathbb{E}[\exp(\lambda X)].$$

We note that the exponential moment assumption is equivalent to the existence of a proper rate function Λ for the lower deviations of S. The lower bound (i) is proved in Section 5.2. The second bound is the rate function of the random walk S evaluated in 0, say $\Lambda(0)$. Indeed, recalling that $\mathbb{E}[X] > 0$, the supremum in the Legendre transform can be taken over $\mathbb{R}^+$ instead of $\mathbb{R}$. Exctracting -1 yields the upper bound above. It is proved in Section 5.3 and used for the proof of the next Corollary 2.3. It can be reached and has a natural interpretation in terms of environmental stochasticity. Indeed, one way to keep the population bounded but alive comes from a 'critical environment', which means $S_n \approx 0$. Then $\mathbb{E}[Z_n \mid \mathcal{E}] = \exp(S_n)$ is neither small nor large and one can expect that the population is positive but bounded. The event $\{S_n \approx 0\}$ is a large deviation event whose probability decreases exponentially with rate $\Lambda(0)$. This bound is thus directly explained by the *environmental stochasticity.*

Now, we focus on the linear fractional case. We derive an explicit expression of ϱ and describe the position of the most recent common ancestor of the population conditioned to be positive but small. We recall that a probability generating function of a random variable R is linear fractional (LF) if there exist positive real numbers m and b such that

$$f(s) = 1 - \frac{1-s}{m^{-1} + bm^{-2}(1-s)/2},$$

where $m = f'(1)$ and $b = f''(1)$. This family includes the probability generating function of geometric distributions, with $b = 2m^2$. More precisely, LF distributions are geometric laws with a second free parameter b which allows to change the probability of the event $\{R = 0\}$.

Corollary 2.3. *We assume that f is a.s. linear fractional, $\mathbb{E}[|X|] < \infty$, $\mathbb{E}[X^2 e^{-X}] < \infty$ and $\mathbb{P}(Z_1 = 0) > 0$.*
We assume also that either $\mathbb{P}(X \geq 0) = 1$ or Assumption 2 hold. Then

$$\varrho \quad = \quad \begin{cases} -\log \mathbb{E}\big[e^{-X}\big] & , \quad \textit{if } \ \mathbb{E}[Xe^{-X}] \geq 0 \\ -\log \inf_{\lambda \geq 0} \mathbb{E}[\exp(\lambda X)] \quad (= \Lambda(0)) & , \quad \textit{else} \end{cases} . \tag{2.1}$$

This result is also stated in the PhD of one of the authors and can be found in [11]. On the level of large deviation (log scale), two regimes in the supercritical case are visible.
If $\mathbb{E}[Xe^{-X}] < 0$, the event $\{Z_n = k\}$ is a typical event in a suitable exceptional environment, say 'critical'. This rare event is then explained (only) by the environmental stochasticity.
If $\mathbb{E}[Xe^{-X}] \geq 0$, we recover a term analogous to the Galton-Watson case, which is smaller than $\Lambda(0)$. The rare event is then due to *demographical stochasticity*.
These two regimes seem to be analogous to the two regimes in the subcritical case, which deal with the asymptotic behavior of $Z_n > 0$, see e.g. [14, 21, 20]. Such regimes for supercritical branching processes have already been observed in [26] in the continuous framework (which essentially represents linear fractional offspring-distributions).

Let us now focus on the most recent common ancestor (MRCA) of the population conditionally on this rare event. More precisely, let $\mathcal{T}^n$ be the set of all ordered, rooted trees of height exactly n. We refer to [31] for classical definitions. We say that an individual i_{n_2} in generation $n_2 > n_1$ stems from an individual i_{n_1} iff there are individuals $i_{n_2-1}, \ldots, i_{n_1+1}$ such that i_{n_2} is a child of i_{n_2-1}, i_{n_2-1} is a child of i_{n_2-2}, ... and i_{n_1+1} is a child of i_{n_1}. Let $T_n \in \mathcal{T}_n$ be the random branching tree, generated by the process $(Z_k)_{0 \leq k \leq n}$ conditioned on $Z_n > 0$ and denote by $MRCA_n$ the most recent common ancestor of the population at time n. More precisely, we consider the events

$$A_k^n := \big\{ \text{ all individuals in generation } n \text{ stem from one individual in generation } n-k \big\}$$

and define the age of the MRCA in generation n as the number of generations one has to go back in the past until all individuals in generation n have a single common ancestor :

$$MRCA_n := \min\{k = 1, 2 \ldots, n \; | A_k^n \; \text{ holds } \}.$$

The case $\mathbb{P}(Z_1 = 0) = 0$ is trivial : extinction is not possible, so $\varrho = -\log \mathbb{P}_1(Z_1 = 1)$ and $\mathbb{P}_2(MRCA_n = n | Z_n = 2) = 1$. It is excluded in the statement below. Moreover, for the sake of simplicity, we restrict the study of the MRCA to starting from one individual and conditioning on $Z_n = 2$.

Corollary 2.4. *We make the same assumptions as in the previous corollary.*
(i) If $\mathbb{E}[Xe^{-X}] < 0$*, then*

$$\liminf_{n\to\infty} \mathbb{P}_1(MRCA_n = n | Z_n = 2) > 0 \qquad ; \qquad \liminf_{n\to\infty} \mathbb{P}_1(MRCA_n = 1 | Z_n = 2) > 0.$$

(ii) If $\mathbb{E}[Xe^{-X}] = 0$*, then for every sequence* $(x_n)_{n\in\mathbb{N}}$ *such that* $x_n \in [1, n]$ *for every* n*, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1(MRCA_n = x_n | Z_n = 2) = 0.$$

Moreover, under the additional assumption $\mathbb{E}\big[f''(1)\big] < \infty$*, there exist two positive finite constants* c, C *such that for every* $n \in \mathbb{N}$*,*

$$c \leq n\ \mathbb{P}_1(MRCA_n = n | Z_n = 2) \leq C$$

and if $\mathbb{E}\big[f''(1)/(1 - f(0))^2\big] < \infty$*, then for every* $\delta \in (0, 1)$*, there exist two positive finite constants* c', C' *such that for every* $n \in \mathbb{N}$*,*

$$c' \leq n^{3/2} \mathbb{P}_1(MRCA_n = \lceil \delta n \rceil | Z_n = 2) \leq C'.$$

(iii) If $\mathbb{E}[Xe^{-X}] > 0$ *and* $\mathbb{E}[e^{(-1-s)X}] < \infty$ *for some* $s > 0$*, then for every* $\delta \in (0, 1]$*,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1(MRCA_n > \delta n | Z_n = 2) < 0.$$

Thus three regimes appear for the most recent common ancestor of the population.
If $\mathbb{E}\big[Xe^{-X}\big] > 0$, which we call the 'strongly' supercritical case, the MRCA is at the end (close to the actual time). The probability that the MRCA is far away from the final generations decreases exponentially. Such a result is classical for branching processes which do not explode, such as subcritical Galton-Watson processes conditioned on survival. It corresponds to a spine decomposition of the population whose subtrees become extinct [25]. Conditionally on $\{Z_n = 2\}$, S is still a random walk with positive drift and will be typically large. Thus the conditioned process is typically small throughout all generations (as in the Galton-Watson case) as growing and then becoming small again within the favorable environment has a very small probability. Consequently, the MRCA will be close to generation n.
But in the 'weakly' supercritical case ($\mathbf{E}\big[Xe^{-X}\big] < 0$), conditionally on $\{Z_n = 2\}$, the MRCA is either at the beginning (close to the root of the tree) or at the end (close to generation n). Such a situation is much less usual. It has already been observed in [16] for the subcritical reduced tree of linear fractional BPRE conditioned to survive. Here, as indicated in the proof of Proposition 2.2 (ii), the random walk S conditioned on $\{Z_n = 2\}$ typically looks like an excursion. It means that S is conditioned on the event $\{\min\{S_0, \dots S_n\} \geq 0, S_n \leq c\}$. In such an environment, subtrees that are either born at the beginning or at the end may survive until the end. All subtrees being born at some generation $\lfloor \delta n \rfloor$, $\delta \in (0, 1)$ experience an unfavorable environment and become extinct. This can be seen as follows. During an excursion from 0 to n, typically $S_{\lfloor \delta n \rfloor} \gg 0$ and thus $e^{S_n - S_{\lfloor \delta n \rfloor}} \ll 0$ and the corresponding subtree will become extinct.
Finally, in the intermediate case ($\mathbb{E}\big[Xe^{-X}\big] = 0$), the MRCA is close to the end, but the probability that the MRCA is far away from the end only decreases polynomially. The intermediately supercritical regime is in-between the two regimes described above and conditioned on $\{Z_n = 2\}$, the typical environment will neither be an excursion nor a random walk with positive drift.
One can expect several more detailed results describing the three regimes, which are beyond the scope of this paper.

3 The Geiger construction for a branching process in varying environment (BPVE)

In this section, we work in a quenched environment, which means that we fix the environment $e := (q_1, q_2, \ldots)$. We consider a branching process in varying environment e and denote by $\mathsf{P}(\cdot)$ (resp. E) the associated probability (resp. expectation), i.e.

$$\mathsf{P}(Z_1 = k_1, \cdots, Z_n = k_n) = \mathbb{P}(Z_1 = k_1, \cdots, Z_n = k_n | \mathcal{E} = e).$$

Thus $(f_1, f_2, \ldots)$ is now fixed and the probability generating function of Z is given by

$$\mathsf{E}\left[s^{Z_n} \mid Z_0 = k\right] = f_{0,n}(s)^k \qquad (0 \leq s \leq 1).$$

We use a construction of Z conditioned on survival, which is due to [18][Proposition 2.1] and extends the spine construction of Galton-Watson processes [25]. In each generation, the individuals are labeled by the integers $i = 1, 2, \cdots$ in a breadth-first manner ('from the left to the right'). We follow then the 'ancestral line' of the leftmost individual having a descendant in generation n. This line is denoted by $\mathbb{L}$. It means that in generation k, the descendance of the individual labeled $\mathbb{L}_k$ survives until time n, whereas all the individuals whose label is less than $\mathbb{L}_k$ become extinct before time n. The Geiger construction ensures that to the left of $\mathbb{L}$, independent subtrees conditioned on extinction in generation n are growing. To the right of $\mathbb{L}$, independent (unconditioned) trees are evolving. Moreover the joint distribution of $\mathbb{L}_k$ and the number of offsprings in generation k is known (see e.g. [1] for details, where $L := \mathbb{L} - 1$) and for every $k \geq 1$, $z \geq 1$ and $1 \leq 1l \leq z$,

$$\mathsf{P}(Z_k = z, \mathbb{L}_k = l | Z_{k-1} = 1, Z_n > 0) = q_k(z) \frac{\mathsf{P}(Z_n > 0 \ |Z_k = 1)\mathsf{P}(Z_n = 0 | Z_k = 1)^{l-1}}{\mathsf{P}(Z_n > 0 \ |Z_{k-1} = 1)}. \tag{3.1}$$

Let us give more details of this construction. We assume that the process starts with $Z_0 = z$ and denote $\mathsf{P}_z(\cdot) := \mathsf{P}(\cdot | Z_0 = z)$. We define for $0 \leq k < n$,

$$\mathsf{p}_{k,n} := \mathsf{P}(Z_n > 0 \mid Z_k = 1) = 1 - f_{k,n}(0), \qquad \mathsf{p}_{n,n} := 1.$$

We can specify the distribution of the number Y_k of unconditioned trees founded by the ancestral line in generation k, at the right of $\mathbb{L}_k$. In generation 0, for $0 \leq i \leq z - 1$,

$$\begin{aligned}\mathsf{P}_z(Y_0 = i | Z_n > 0) := \mathsf{P}_z(\mathbb{L}_0 = z - i \mid Z_n > 0) &= \frac{\mathsf{P}(Z_n > 0 \ |Z_0 = 1)\mathsf{P}(Z_n = 0 | Z_0 = 1)^{z-i-1}}{\mathsf{P}(Z_n > 0 \ |Z_0 = z)} \\ &= \frac{1 - f_{0,n}(0)}{\mathsf{P}_z(Z_n > 0)} f_{0,n}(0)^{z-i-1}. \end{aligned} \tag{3.2}$$

More generally, for all $1 \leq k \leq n$ and $i \geq 0$, (3.1) yields

$$\begin{aligned}\mathsf{P}(Y_k = i | Z_n > 0) &:= \mathsf{P}(Z_k - \mathbb{L}_k = i | Z_n > 0, Z_{k-1} = 1) \\ &= \sum_{j=i+1}^{\infty} \mathsf{P}(Z_k = j, \mathbb{L}_k = j - i | Z_n > 0, Z_{k-1} = 1) \\ &= \frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}} \sum_{j=i+1}^{\infty} q_k(j) f_{k,n}(0)^{j-i-1}. \end{aligned} \tag{3.3}$$

Finally, we note that $f_{n,n}(0) = 0$, thus for $k = n$, we have $\mathsf{P}(Y_n = i | Z_n > 0) = \frac{q_n(i+1)}{p_{n-1,n}}$.

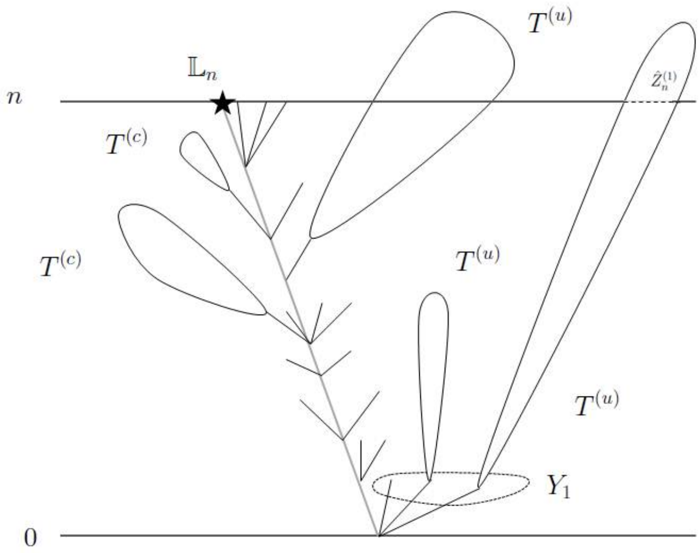

Figure 1: Geiger construction with $T^{(c)}$ trees conditioned on extinction and $T^{(u)}$ unconditioned trees.

Here, we do not require the full description of the conditioned tree since we are only interested in the population alive at time n. Thus we do not have to consider the trees conditioned on extinction, which grow to the left of $\mathbb{L}$. We can construct the population alive in generation n using the i.i.d random variables $\hat{Y}_0, \hat{Y}_1, \hat{Y}_2, \ldots, \hat{Y}_n$ whose distribution is specified by (3.2) and (3.3) :

$$\mathbb{P}(\hat{Y}_k = i) = \mathsf{P}(Y_k = i | Z_n > 0)$$

Let $(\hat{Z}_j^{(k)})_{j \geq 0}$ be independent branching processes in varying environment which are distributed as Z for $j > k$ and satisfy

$$\hat{Z}_j^{(k)} := 0 \text{ for } \quad j < k, \qquad \hat{Z}_k^{(k)} := \hat{Y}_k.$$

More precisely, for all $0 \leq k \leq n$ and $z_0, \cdots, z_n \geq 0$,

$$\begin{aligned}\mathsf{P}(\hat{Z}_0^{(k)} = 0, \cdots, \hat{Z}_{k-1}^{(k)} = 0, \hat{Z}_k^{(k)} = z_k, \hat{Z}_{k+1}^{(k)} = z_{k+1}, \cdots, \hat{Z}_n^{(k)} = z_n)& \\ = \mathsf{P}(\hat{Y}_k = z_k)\mathsf{P}(Z_{k+1} = z_{k+1}, \cdots, Z_n = z_n \mid Z_k = z_k).&\end{aligned}$$

The sizes of the independent subtrees generated by the ancestral line in generation k, which may survive until generation n, are given by $(\hat{Z}_j^{(k)})_{0 \leq j \leq n}$, $0 \leq k \leq n-1$. In particular,

$$\mathcal{L}(Z_n | Z_n > 0) = \mathcal{L}(\hat{Z}_n^{(0)} + \ldots + \hat{Z}_n^{(n-1)} + \hat{Y}_n + 1). \tag{3.4}$$

Lemma 3.1. *The probability that all subtrees emerging before generation n become extinct before generation n is given for $z \geq 1$ by*

$$\mathsf{P}_z(\hat{Z}_n^{(0)} + \ldots + \hat{Z}_n^{(n-1)} = 0) = \prod_{k=0}^{n-1} \mathsf{P}_z(\hat{Z}_n^{(k)} = 0) = \frac{\mathsf{p}_{n-1,n}}{\mathsf{p}_{-1,n}} \prod_{k=0}^{n-1} f_k'(f_{k,n}(0)),$$

where we use the following convenient notation $f_0(s) := s^z$, $p_{-1,n} := \mathsf{P}_z(Z_n > 0)$.

Proof. First, we compute the probability that the subtree generated by the ancestral line in generation k does not survive until generation n, i.e. $\{\hat{Z}_n^{(k)} = 0\}$. By (3.3), for $k \geq 1$,

$$\begin{aligned}
\mathsf{P}(\hat{Z}_n^{(k)} = 0) &= \sum_{i=0}^{\infty} \mathsf{P}_z(\hat{Y}_k = i)\mathsf{P}(Z_n = 0 | Z_k = i) \\
&= \frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}} \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} q_k(j) f_{k,n}(0)^{j-i-1} \cdot f_{k,n}(0)^i \\
&= \frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}} \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} q_k(j) f_{k,n}(0)^{j-1} \\
&= \frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}} \sum_{j=1}^{\infty} j q_k(j) f_{k,n}(0)^{j-1} \\
&= \frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}} f_k'(f_{k,n}(0)). \qquad (3.5)
\end{aligned}$$

Similarly, we get from (3.2) that

$$\begin{aligned}
\mathsf{P}_z(\hat{Z}_n^{(0)} = 0) &= \sum_{i=0}^{z-1} \mathsf{P}_z(Y_0 = i | Z_n > 0)\mathsf{P}(Z_n = 0 | Z_0 = i) \\
&= \sum_{i=0}^{z-1} \frac{1 - f_{0,n}(0)}{\mathsf{P}_z(Z_n > 0)} f_{0,n}(0)^{z-i-1} f_{0,n}(0)^i \\
&= \frac{\mathsf{p}_{0,n}}{\mathsf{p}_{-1,n}} z f_{0,n}(0)^{z-1} = \frac{\mathsf{p}_{0,n}}{\mathsf{p}_{-1,n}} f_0'(f_{0,n}(0))
\end{aligned}$$

with the convention $f_0(s) = s^z$. Adding that the subtrees given by $(\hat{Z}_j^{(k)})_{j \geq 0}$ are independent, a telescope argument yields the claim. □

For the next lemma, we introduce the last generation before n when the environment allows extinction :

$$\kappa_n := \sup\{1 \leq k \leq n \ : \ q_k(0) > 0\}, \qquad (\sup \varnothing = 0).$$

Note that κ_n only depends on the environment up to generation n.

Lemma 3.2. *Let* z_0 *be the smallest element in* $\mathcal{I}$. *Then,*

$$\mathsf{P}_{z_0}(Z_n = z_0 \mid Z_n > 0) = \frac{q_{\kappa_n}(z_0)}{\mathsf{p}_{\kappa_n - 1, \kappa_n}} \times \prod_{k=0}^{\kappa_n - 1} \frac{\mathsf{p}_{k,\kappa_n}}{\mathsf{p}_{k-1,\kappa_n}} f_k'(f_{k,\kappa_n}(0)) \times \prod_{j=\kappa_n+1}^{n} q_j(1)^{z_0},$$

where we recall the following convenient notation $f_0(s) = s^z$, $p_{-1,n} = \mathsf{P}_z(Z_n > 0)$.

Proof. By definition of $\mathcal{I}$ and z_0, $q(0) > 0$ implies $q(k) = 0$ for every $1 \leq k < z_0$. We first deal with the case $\kappa_n > 0$. Then,

$$q_{\kappa_n}(0) > 0, \quad q_{\kappa_n}(k) = 0 \text{ if } 1 \leq k < z_0 \ ; \qquad q_{\kappa_n+1}(0) = \cdots = q_n(0) = 0.$$

In particular the number of individuals in generation κ_n is at least z_0 times the number of individuals in generation $\kappa_n - 1$ who leave at least one offspring in generation κ_n. Moreover, as extinction is not possible after generation κ_n, it holds that $Z_{\kappa_n} \leq Z_{\kappa_n+1} \leq \cdots \leq Z_n$.
Let us consider the event $Z_n = z_0 > 0$. Then $Z_{\kappa_n - 1} > 0$ and $Z_{\kappa_n} \geq z_0$. So $Z_{\kappa_n} = Z_{\kappa_n+1} =$

$\cdots = Z_n = z_0$ and only a single individual in generation $\kappa_n - 1$ leaves one offspring (or more) in generation κ_n. This individual lives on the ancestral line. Thus all the subtrees to the right of the ancestral line which are born before generation κ_n have become extinct before generation κ_n, i.e. $\hat{Z}^{(0)}_{\kappa_n} = \ldots = \hat{Z}^{(\kappa_n-1)}_{\kappa_n} = 0$. In generation $\kappa_n - 1$, the individual on the ancestral line has z_0 offsprings and $\hat{Y}_{\kappa_n} = z_0 - 1$. After generation κ_n, all the individuals must leave exactly one offspring to keep the population constant until generation n, since $q_{\kappa_n+1}(0) = \cdots = q_n(0) = 0$. This probability is then given by $q_j(1)^{z_0}$ in generation $j > \kappa_n$. Moreover, (3.3) simplifies to $\mathsf{P}(\hat{Y}_{\kappa_n} = z_0 - 1) = q_{\kappa_n}(z_0)/\mathsf{p}_{\kappa_n-1,\kappa_n}$. Using the previous lemma, it can be written as follows:

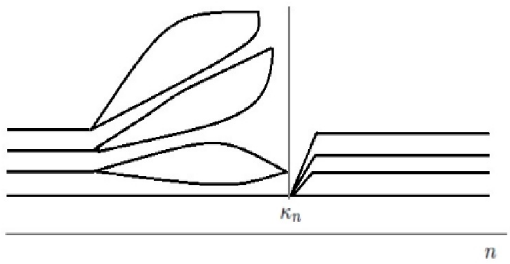

Figure 2: Illustration of the proof of Lemma 3.2.

$$
\begin{aligned}
&\mathsf{P}_{z_0}(Z_n = z_0) \\
&\quad = \mathsf{P}_{z_0}(\hat{Z}^{(0)}_{\kappa_n} = \ldots = \hat{Z}^{(\kappa_n-1)}_{\kappa_n} = 0)\mathsf{P}(\hat{Y}_{\kappa_n} = z_0 - 1)\mathsf{P}_{z_0}(\hat{Z}^{(\kappa_n)}_n + \ldots + \hat{Z}^{(n-1)}_n + Y_n + 1 = z_0) \\
&\quad = \left[\prod_{k=0}^{\kappa_n-1} \frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}} f'_k(f_{k,\kappa_n}(0))\right] \frac{q_{\kappa_n}(z_0)}{\mathsf{p}_{\kappa_n-1,\kappa_n}} \mathsf{P}_{z_0}(\hat{Z}^{(\kappa_n)}_n + \ldots + \hat{Z}^{(n-1)}_n + Y_n + 1 = z_0) \\
&\quad = \frac{q_{\kappa_n}(z_0)}{\mathsf{p}_{\kappa_n-1,\kappa_n}} \left[\prod_{k=0}^{\kappa_n-1} \frac{\mathsf{p}_{k,n}}{\mathsf{p}_{k-1,n}} f'_k(f_{k,\kappa_n}(0))\right] \prod_{j=\kappa_n+1}^{n} q_j(1)^{z_0}.
\end{aligned}
$$

Recall that after generation κ_n, each individual has at least one offspring and thus $\mathsf{p}_{j,n} = \mathsf{p}_{j,\kappa_n}$ for any $j < \kappa_n$. This ends up the proof in the case $\kappa_n > 0$. The case when $\kappa_n = 0$ is easier. Indeed,

$$\mathsf{P}_{z_0}(Z_n = z_0) = \mathsf{P}_{z_0}(Z_1 = \cdots = Z_n = z_0) = \prod_{j=1}^{n} q_j(1)^{z_0}$$

since $q_{\kappa_n+1}(0) = \cdots = q_n(0) = 0$ and Z is nondecreasing until generation n. $\square$

4 Proof of Theorem 2.1 : the probability of staying positive but bounded

In this section, we prove Theorem 2.1 with the help of two lemmas. The first lemma establishes the existence of a proper 'common' limit.

Lemma 4.1. *Assume that $z \geq 1$ satisfies $\mathbb{P}(Q(0) > 0, Q(z) > 0) > 0$.*
Then for all $k, j \in Cl(\{z\})$, the following limits exist in $[0, \infty)$ and coincide

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_k(Z_n = j) = \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(Z_n = z).$$

Moreover, for every sequence k_n such that $k_n \geq z$ for n large enough and $k_n/n \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(Z_n = z) = \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(1 \leq Z_n \leq k_n).$$

Proof. Note that for every $k \geq 1$, $\mathbb{P}_k(Z_1 = z) > 0$ since

$$\mathbb{P}_k\big(Z_1 = z \mid Q_1\big) \geq Q_1(0)^{k-1} Q_1(z), \qquad \mathbb{P}(Q(0) > 0, Q(z) > 0) > 0.$$

We know that by Markov property, for all $m, n \geq 1$,

$$\mathbb{P}_z(Z_{n+m} = z) \geq \mathbb{P}_z(Z_n = z)\mathbb{P}_z(Z_m = z). \tag{4.1}$$

Adding that $\mathbb{P}_z(Z_1 = z) > 0$, we obtain that the sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n := -\log \mathbb{P}_z(Z_n = z)$ is finite and subadditive. Then Fekete's lemma ensures that $\lim_{n\to\infty} a_n/n$ exists and belongs to $[0, \infty)$. Next, if $j, k \in Cl(\{z\})$, there exist $l, m \geq 0$ such that $\mathbb{P}_z(Z_l = j) > 0$ and $\mathbb{P}_z(Z_m = k) > 0$. We get

$$\mathbb{P}_k(Z_{n+l+1} = j) \geq \mathbb{P}_k(Z_1 = z)\mathbb{P}_z(Z_n = z)\mathbb{P}_z(Z_l = j)$$

and

$$\mathbb{P}_z(Z_{m+n+1} = z) \geq \mathbb{P}_z(Z_m = k)\mathbb{P}_k(Z_n = j)\mathbb{P}_j(Z_1 = z).$$

Adding that $\mathbb{P}_j(Z_1 = z) > 0$, we obtain

$$\liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_k(Z_n = j) \geq \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(Z_n = z) \geq \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_k(Z_n = j),$$

which yields the first result.

For the second part of the lemma, we first observe that $\mathbb{P}_z(Z_n = z) \leq \mathbb{P}_z(1 \leq Z_n \leq k_n)$ for n large enough. To prove the converse inequality, we define for $\varepsilon > 0$ the set

$$\mathcal{A}_\varepsilon := \{q \in \Delta | q(0) > \varepsilon,\, q(z) > \varepsilon\}.$$

According to the definition of $\mathcal{I}$ and the assumption, $\mathbb{P}(Q \in \mathcal{A}_\varepsilon) > 0$ if ε is chosen small enough. Thus, we get

$$\begin{aligned}
\mathbb{P}_z(Z_n = z) &\geq \mathbb{P}_z(1 \leq Z_{n-1} \leq k_n) \min_{1\leq j\leq k_n} \mathbb{P}_j(Z_1 = z) \\
&\geq \mathbb{P}_z(1 \leq Z_{n-1} \leq k_n)\mathbb{P}(Q \in \mathcal{A}_\varepsilon) \min_{1\leq j\leq k_n} \mathbb{E}[\mathbb{P}_1(Z_1 = z)\mathbb{P}_1(Z_1 = 0|Q)^{j-1}|Q \in \mathcal{A}_\varepsilon] \\
&\geq \mathbb{P}_z(1 \leq Z_{n-1} \leq k_n)\mathbb{P}(Q \in \mathcal{A}_\varepsilon)\, \varepsilon^{k_n}.
\end{aligned}$$

Using the logarithm of this expression and letting $n \to \infty$ yields

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_z(Z_n = z) \geq \limsup_{n\to\infty} \left(\tfrac{1}{n} \log \mathbb{P}_z(1 \leq Z_{n-1} \leq k_n) + \log(\varepsilon)\tfrac{k_n}{n}\right).$$

Adding that $k_n = o(n)$ by assumption gives the claim. $\square$

Now, we prove a representation of the limit ρ in terms of generating functions. It will be useful in the rest of the paper.

Lemma 4.2. *Assume that $\mathbb{P}_1(Z_1 = 0) > 0$. Then for all $i, j \in Cl(\mathcal{I})$,*

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_i(Z_n = j) = \lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}\Big[Q_n(z_0) f_{0,n}(0)^{z_0-1} \prod_{i=1}^{n-1} f_i'\big(f_{i,n}(0)\big)\Big],$$

where z_0 is the smallest element in $\mathcal{I}$.

We note that $\mathbb{P}_1(Z_1 = 0) > 0$ is equivalent to $\mathbb{P}(Q(0) > 0) > 0$ and in view of Lemma 4.1, we only have to prove the result for $k = j = z_0$, where z_0 is the smallest element in $\mathcal{I}$. Differentiation of the probability generating function of Z_n yields the result for $z_0 = 1$. The generalization of the result for $z_0 \neq 1$ via higher order derivatives of generating functions appears to be complicated. Instead, we use probabilistic arguments, involving the Geiger construction of the previous section.

Proof. First, the result is obvious when $z_0 = 1 \in \mathcal{I}$ since

$$\mathbb{P}(Z_n = 1|\mathcal{E}) = \frac{d}{ds} f_{0,n}(s)\big|_{s=0} = f_n'(0) \cdot \prod_{i=1}^{n-1} f_i'(f_{i,n}(0)).$$

For the case $z_0 > 1$, we start by proving the lower bound. Using (3.4), Lemma 3.1, (3.2) and recalling that $\mathbb{P}_{z_0}(Z_n > 0|\mathcal{E}) = p_{-1,n}$, we have

$$\begin{aligned}
\mathbb{P}_{z_0}(Z_n = z_0) &= \mathbb{E}\Big[\mathbb{P}_{z_0}(Z_n = z_0 | Z_n > 0, \mathcal{E})\mathbb{P}_{z_0}(Z_n > 0|\mathcal{E})\Big] \\
&= \mathbb{E}\Big[\mathbb{P}_{z_0}(\hat{Z}_n^{(0)} + \cdots + \hat{Z}_n^{(n-1)} + \hat{Y}_n + 1 = z_0|\mathcal{E})\mathbb{P}_{z_0}(Z_n > 0|\mathcal{E})\Big] \\
&\geq \mathbb{E}\Big[\mathbb{P}_{z_0}(\hat{Z}_n^{(0)} + \cdots + \hat{Z}_n^{(n-1)} = 0, \hat{Y}_n = z_0 - 1|\mathcal{E})\mathbb{P}_{z_0}(Z_n > 0|\mathcal{E})\Big] \\
&= \mathbb{E}\Big[\mathbb{P}_{z_0}(Z_n > 0|\mathcal{E})\ \frac{Q_n(z_0)}{\mathsf{p}_{n-1,n}}\ \frac{\mathsf{p}_{n-1,n}}{\mathsf{p}_{-1,n}} \prod_{i=0}^{n-1} f_i'(f_{i,n}(0))\Big] \\
&= \mathbb{E}\Big[Q_n(z_0) \prod_{i=0}^{n-1} f_i'\big(f_{i,n}(0)\big)\Big].
\end{aligned} \tag{4.2}$$

Recalling also that $f_0'(s) = z_0 s^{z_0 - 1}$, we get

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_{z_0}(Z_n = z_0) \geq \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}\Big[Q_n(z_0) f_{0,n}^{z_0-1}(0) \prod_{i=1}^{n} f_i'\big(f_{i,n}(0)\big)\Big].$$

Let us now prove the converse inequality. Following the previous section, z_0 is the smallest element in $\mathcal{I}$ and κ_n is the (now random) last moment when an environment satifies $Q(0) > 0$. We decompose the event $\{Z_n = z_0\}$ according to κ_n :

$$\mathbb{P}_{z_0}(Z_n = z_0) = \sum_{k=0}^{n} \mathbb{E}\Big[\mathbb{P}_{z_0}(Z_n = z_0|\mathcal{E}, Z_n > 0)\mathbb{P}_{z_0}(Z_n > 0|\mathcal{E}); \kappa_n = k\Big]$$

Using that conditionally on $\kappa_n = k$, $\mathbb{P}_{z_0}(Z_n > 0|\mathcal{E}) = \mathbb{P}_{z_0}(Z_k > 0|\mathcal{E})$ and Lemma 3.2, we get by independence

$$\begin{aligned}
\mathbb{P}_{z_0}(Z_n = z_0) &= \sum_{k=0}^{n} \mathbb{E}\Big[\mathbb{P}_{z_0}(Z_k > 0|\mathcal{E}) \frac{Q_k(z_0)}{\mathsf{p}_{k-1,k}} \frac{\mathsf{p}_{k-1,k}}{\mathsf{p}_{-1,k}} \prod_{i=0}^{k-1} f_i'(f_{i,k}(0)) \prod_{j=k+1}^{n} Q_j(1)^{z_0}\ ;\ \kappa_n = k\Big] \\
&\leq \sum_{k=0}^{n} \mathbb{E}\Big[Q_k(z_0) \prod_{i=0}^{k-1} f_i'(f_{i,k}(0))\Big] \prod_{j=k+1}^{n} \mathbb{E}\big[Q_j(1)^{z_0}\big] \\
&= \sum_{k=1}^{n} \mathbb{E}\Big[Q_k(z_0) \prod_{i=0}^{k-1} f_i'(f_{i,k}(0))\Big] \mathbb{E}\big[Q(1)^{z_0}\big]^{n-k-1} + \mathbb{E}\big[Q(1)^{z_0}\big]^{n}.
\end{aligned}$$

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+$ and $b > 0$. Then, by standard results on the exponential rate of sums, we have

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \sum_{k=1}^{n} a_k b^{n-k} = \max\Big\{\limsup_{n\to\infty} \tfrac{1}{n} \log a_n, \log b\Big\}.$$

Thus

$$\begin{aligned}
&\lim_{n\to\infty} \tfrac{1}{n}\log \mathbb{P}_{z_0}(Z_n = z_0)\\
&\qquad \le \max\Big\{ \limsup_{n\to\infty} \tfrac{1}{n}\log \mathbb{E}\big[Q_n(z_0)\prod_{i=0}^{n} f_i'(f_{i,n}(0))\big]; \log \mathbb{E}\big[Q(1)^{z_0}\big]\Big\}.
\end{aligned}$$

We now prove that the first term always realizes the maximum. Using that $f_0'(f_{0,n}(0)) = z_0 f_{0,n}^{z_0-1}(0) = z_0\mathbb{P}_1(Z_n = 0|\mathcal{E})^{z_0-1}$ and $f_i'\big(f_{i,n}(0)\big) \ge f'(0)$, we have

$$\begin{aligned}
\mathbb{E}\Big[Q_n(z_0)\prod_{i=0}^{n-1} f_i'(0)\Big] &\ge z_0\mathbb{E}\Big[Q_n(z_0)\mathbb{P}_1(Z_n = 0|\mathcal{E})^{z_0-1}\prod_{i=1}^{n-1} Q_i(1)\Big]\\
&\ge z_0\mathbb{E}\Big[Q_n(z_0)\Big(Q_n(0)\prod_{i=1}^{n-1} Q_i(1)\Big)^{z_0-1}\prod_{i=1}^{n-1} Q_i(1)\Big]\\
&\ge z_0\mathbb{E}\Big[Q_n(z_0)Q_n(0)^{z_0-1}\Big]\mathbb{E}\big[Q(1)^{z_0}\big]^n,
\end{aligned}$$

By definition of z_0, $\mathbb{E}\big[Q_n(z_0)Q_n(0)^{z_0-1}\big] > 0$, and we can conclude

$$\lim_{n\to\infty} \tfrac{1}{n}\log \mathbb{P}_{z_0}(Z_n = z_0) \le \limsup_{n\to\infty} \tfrac{1}{n}\log \mathbb{E}\big[Q_n(z_0)f_{0,n}(0)^{z_0-1}\prod_{i=1}^{n} f_i'(f_{i,n}(0))\big]$$

We end up the proof by checking the convergence of the sequence on the right-hand side above. We use again (3.4) and Lemma 3.1 to write

$$\begin{aligned}
\phi_n :&= \mathbb{P}_{z_0}(\hat{Z}_n^{(0)} + \ldots + \hat{Z}_n^{(n-1)} = 0, Z_n = z_0)\\
&= \mathbb{E}\Big[\mathbb{P}_{z_0}(\hat{Z}_n^{(0)} + \ldots + \hat{Z}_n^{(n-1)} = 0\ |\mathcal{E})\mathbb{P}(\hat{Y}_n = z_0 - 1)\Big]\\
&= \mathbb{E}\Big[Q_n(z_0)\prod_{i=0}^{n} f_i'\big(f_{i,n}(0)\big)\Big]. \qquad (4.3)
\end{aligned}$$

It is the probability of having z_0-many individuals in generation n, where all individuals in generation n have a common ancestor in generation $n-1$. By Markov property, for $k = 1, \ldots, n$

$$\begin{aligned}
&\mathbb{P}_{z_0}(\hat{Z}_n^{(0)} + \ldots + \hat{Z}_n^{(n-1)} = 0, Z_n = z_0)\\
&\quad \ge \mathbb{P}_{z_0}(\hat{Z}_k^{(0)} + \ldots + \hat{Z}_k^{(k-1)} = 0, Z_k = z_0)\mathbb{P}_{z_0}(\hat{Z}_{n-k}^{(0)} + \ldots + \hat{Z}_{n-k}^{(n-k-1)} = 0, Z_{n-k} = z_0).
\end{aligned}$$

The same subadditivity arguments as in the proof of Lemma 4.1 applied to ϕ_n yield the existence of the limit of $\frac{1}{n}\log\phi_n$ and

$$\lim_{n\to\infty} \tfrac{1}{n}\log \mathbb{P}_{z_0}(Z_n = z_0) = \lim_{n\to\infty} \tfrac{1}{n}\log\phi_n = \lim_{n\to\infty} \tfrac{1}{n}\log \mathbb{E}\Big[Q_n(z_0)f_{0,n}(0)^{z_0-1}\prod_{i=1}^{n} f_i'\big(f_{i,n}(0)\big)\Big] \quad (4.4)$$

This ends up the proof.

□

5 Proof of Proposition 2.2

5.1 Preliminaries on random walks

In this section, we will shortly present some standard results on random walks which we use. In all following results, we take $S_0 = 0$. We assume that there exists $s > 0$ such that $\mathbb{E}[Xe^{-sX}] = 0$.

This suggests to change to a measure $\mathbf{P}$, defined by

$$\mathbf{P}(X \in dx) := \frac{e^{-sx}\mathbb{P}(X \in dx)}{\mu},$$

where $\mu := \mathbb{E}[e^{-sX}]$. We note that $\mathbf{E}[X] = \mu^{-1}\mathbb{E}[Xe^{-sX}] = 0$, and that S is a recurrent random walk under $\mathbf{P}$. In the following proofs, we use the change of measure described here. We define

$$L_n := \min\{S_1, \ldots, S_n\}$$

and

$$M_n := \min\{S_1, \ldots, S_n\}.$$

The following result is directly derived from [2][Proposition 2.1]. For lattice random walks, the claims result e.g. from [34][Theorem 6].

Lemma 5.1. *Assume that* $\mathbf{E}[X] = 0$ *and* $\mathbf{Var}(X) < \infty$. *Then for every* $\theta > 0$, *there exists* $d = d(\theta)$ *such that*

$$\mathbf{E}[e^{-\theta S_n}; L_n \geq 0] \sim d\, n^{-\frac{3}{2}} \quad (n \to \infty)$$

and

$$\mathbf{E}[e^{\theta S_n}; M_n < 0] \sim d\, n^{-\frac{3}{2}} \quad (n \to \infty).$$

The following lemma results from [2][Equation (2.5) therein].

Lemma 5.2. *Assume that* $\mathbf{E}[X] = 0$ *and* $\mathbf{Var}(X) < \infty$. *Then for every* $c > 0$ *large enough, there exists* $d = d(c)$ *such that*

$$\mathbf{P}(L_n \geq 0, S_n \leq c) \sim d\, n^{-\frac{3}{2}} \qquad (n \to \infty).$$

Remark: In the non-lattice case, the result holds for every $c > 0$. In the lattice case, c must be chosen such that $\mathbf{P}(0 \leq S_1 \leq c) > 0$.

From the previous results, it follows that

Corollary 5.3. *Assume that* $\mathbf{E}[X] = 0$ *and* $\mathbf{Var}(X) < \infty$. *Then for every* $\theta > 1$,

$$\mathbf{E}\big[e^{-S_n + \theta L_n}\big] = O(n^{-3/2}).$$

Proof. We use a decomposition according to the first minimum of the random walk, i.e. let

$$\tau_n := \min\{k \in \{0, \ldots, n\} \mid S_k = L_n \wedge 0\}.$$

Decomposing at the first minimum and using duality yields

$$\begin{aligned}
\mathbf{E}\big[e^{-S_n + \theta L_n}\big] &= \sum_{k=0}^{n} \mathbf{E}\big[e^{-(S_n - \theta S_k)}; \tau_n = k\big] \\
&= \sum_{k=0}^{n} \mathbf{E}\big[e^{-(S_n - S_k)} e^{(\theta-1)S_k}; \tau_k = k, \min_{i=k,\ldots,n}\{S_n - S_i\} \geq 0\big] \\
&= \sum_{k=0}^{n} \mathbf{E}\big[e^{(\theta-1)S_k}; \tau_k = k\big]\mathbf{E}\big[e^{-S_{n-k}}; L_{n-k} \geq 0\big] \\
&= \sum_{k=0}^{n} \mathbf{E}\big[e^{(\theta-1)S_k}; M_k < 0\big]\mathbf{E}\big[e^{-S_{n-k}}; L_{n-k} \geq 0\big].
\end{aligned}$$

The last step follows from duality (see e.g. [1]). Recall that by assumption, $\theta - 1 > 0$. Applying Lemma 5.1 now yields that there is a $c < \infty$ such that for n large enough

$$\begin{aligned}\mathbf{E}\big[e^{-S_n+\theta L_n}\big] &\leq c\Big(\frac{1}{n^{3/2}} + \sum_{k=1}^{n-1} \frac{1}{(n-k)^{3/2}k^{3/2}}\Big)\\ &\leq c\Big(\frac{2d}{n^{3/2}} + \frac{2d}{\lfloor n/2\rfloor^{3/2}\sum_{k=0}^{\lfloor n/2\rfloor} k^{3/2}}\Big) = O(n^{-3/2}),\end{aligned}$$

which is the claim of the corollary. $\square$

In the next lemma, we will use probability measures $\mathbf{P}^+$ and $\mathbf{P}^-$ which e.g. have been introduced in [2]. Here, we recall the definition. Define the renewal functions $u : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} u(x) &= 1 + \sum_{k=1}^{\infty} \mathbf{P}(-S_k \leq x, \max\{S_1, \dots, S_k\} < 0), \quad x \geq 0,\\ v(x) &:= 1 + \sum_{k=1}^{\infty} \mathbf{P}(-S_k > x, \min\{S_1, \dots, S_k\} \geq 0), \quad x \leq 0\ ,\\ v(0) &= u(0) = 1,\end{aligned}$$

and 0 elsewhere. Using the identities

$$\begin{aligned}\mathbf{E}[u(x+X); X + x \geq 0] &= u(x), \quad x \geq 0,\\ \mathbf{E}[v(x+X); X + x < 0] &= v(x), \quad x \leq 0,\end{aligned} \tag{5.1}$$

which hold for every oscillating random walk, one can construct probability measures $\mathbf{P}^+$ and $\mathbf{P}^-$: Define the filtration $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$, where $\mathcal{F}_n = \sigma(Q_1, \dots, Q_n, Z_0, \dots, Z_n)$. Then S is adapted to $\mathcal{F}$ and X_{n+1} (as well as Q_{n+1}) is independent of $\mathcal{F}_n$ for all $n \geq 0$. Now, for every bounded, $\mathcal{F}_n$-measurable random variable R_n we can define

$$\begin{aligned}\mathbf{E}_x^+[R_n] &= \frac{1}{u(x)}\mathbf{E}_x[R_n u(S_n); L_n \geq 0], \quad x \geq 0,\\ \mathbf{E}_x^-[R_n] &= \frac{1}{v(x)}\mathbf{E}_x[R_n v(S_n); M_n < 0], \quad x \leq 0.\end{aligned}$$

The probability measures $\mathbf{P}_x^+$ and $\mathbf{P}_x^-$ correspond to conditioning the random walk S on not to enter $(-\infty, 0)$ and $[0, \infty)$ respectively.
More precisely, if $R_n \to R$ a.s. with respect to $\mathbf{P}^+$ (resp. $\mathbf{P}^-$), then

$$\begin{aligned}&\lim_{n\to\infty} \mathbf{P}(R_n \in \cdot | L_n \geq 0) \to \mathbf{P}^+(R \in \cdot)\\ &\lim_{n\to\infty} \mathbf{P}(R_n \in \cdot | M_n < 0) \to \mathbf{P}^-(R \in \cdot).\end{aligned}$$

The first result is proved in [4][Lemma 2.5] in the more general setting of random walks in the domain of attraction of a stable law. The proof of the second claim is analogous.
We end up with an asymptotic result in the critical case, which is stated and proved only in the non-lattice case.

Lemma 5.4. *We assume that* $\mathbf{E}[X] = 0$, $\mathbf{Var}(X) < \infty$, *and that* $\mathbf{E}\big[(\log^+ \xi_Q(a))^{2+\varepsilon}\big] < \infty$ *for some* $\varepsilon > 0$. *Then, for every* $c > 0$,

$$\liminf_{n\to\infty} \mathbf{P}(Z_n > 0 | L_n \geq 0, S_n \leq c) > 0.$$

Proof. The proof follows essentially [2] and we just present the main steps. First, Lemmas 5.1 and 5.2 ensure that for all $\theta, c > 0$ large enough, there exists $d > 0$ such that

$$\mathbf{E}\big[e^{-\theta S_n}; L_n \geq 0\big] \sim d\ \mathbf{P}(L_n \geq 0, S_n \leq c) \qquad (n \to \infty). \tag{5.2}$$

Secondly, we recall the well-known estimate (see e.g. [5][Lemma 2])

$$\mathbf{P}(Z_n > 0 \mid \mathcal{E}) \geq \frac{1}{e^{-S_n} + \sum_{i=0}^{n-1} \eta_{i+1} e^{-S_i}} \quad \text{a.s.,}$$

where $\eta_i := \sum_{y=1}^{\infty} y(y-1)Q_i(y)/m_{Q_i}^2$. Then, we rewrite

$$\begin{aligned}
&\mathbf{E}\Big[\frac{1}{e^{-S_n} + \sum_{i=0}^{n-1} \eta_{i+1} e^{-S_i}}; L_n \geq 0, S_n \leq c\Big] \\
&\quad \geq \mathbf{E}\Big[\frac{1}{1 + \sum_{i=0}^{\lfloor n/2 \rfloor} \eta_{i+1} e^{-S_i} + e^{-S_{\lfloor n/2 \rfloor}} \sum_{i=\lfloor n/2 \rfloor+1}^{n-1} \eta_{i+1} e^{S_{\lfloor n/2 \rfloor} - S_i}}; L_n \geq 0, S_n \leq c\Big] \\
&\quad \geq \mathbf{E}\Big[\frac{(c - S_n)^+ \wedge 1}{1 + \sum_{i=0}^{\lfloor n/2 \rfloor} \eta_{i+1} e^{-S_i} + \sum_{i=\lfloor n/2 \rfloor+1}^{n-1} \eta_{i+1} e^{S_{\lfloor n/2 \rfloor} - S_i}}; L_n \geq 0\Big] \\
&\quad = \mathbf{E}\Big[\varphi(U_n, \tilde{V}_n, S_n); L_n \geq 0\Big] \\
&\quad \geq e^{-c/2} \mathbf{E}\Big[e^{-S_n/2} \varphi(U_n, \tilde{V}_n, S_n); L_n \geq 0\Big],
\end{aligned}$$

where $U_n := \sum_{i=0}^{\lfloor n/2 \rfloor} \eta_{i+1} e^{-S_i}$, $\tilde{V}_n := \sum_{i=\lfloor n/2 \rfloor+1}^{n-1} \eta_{i+1} e^{S_{\lfloor n/2 \rfloor} - S_i}$ and $\varphi(u, v, z) = (1 + u + v)^{-1}(c - z)^+ \wedge 1$. Using (5.2), it becomes (with $\theta = \frac{1}{2}$)

$$\liminf_{n \to \infty} \mathbf{P}(Z_n > 0 | L_n \geq 0, S_n \leq c) \geq d^{-1} \liminf_{n \to \infty} \frac{e^{-c/2} \mathbf{E}\Big[e^{-S_n/2} \varphi(U_n, \tilde{V}_n, S_n); L_n \geq 0\Big]}{\mathbf{E}\big[e^{-S_n/2}; L_n \geq 0\big]}.$$

Due to monotonicity and Lemma 3.1 in [2], the limits of $U_\infty = \lim_{n \to \infty} U_n$ and $V_\infty = \lim_{n \to \infty} \sum_{i=0}^{\lfloor n/2 \rfloor} \eta_i e^{S_i}$ exist and are finite respectively under the probabilities $\mathbf{P}^+$-a.s. and $\mathbf{P}^-$-a.s. Thus all conditions of Proposition 2.5 in [2] are met. Applying this proposition with $\theta = 1/2$, there exists a non zero measure $\nu_{1/2}$ on $\mathbb{R}^+$ which gives the convergence of the right-hand side above and

$$\liminf_{n \to \infty} \mathbf{P}(Z_n > 0 | L_n \geq 0, S_n \leq c) \quad \geq \int_{\mathbb{R}_+^3} \varphi(u, v, -z) \mathbf{P}^+(U_\infty \in du) \mathbf{P}^-(V_\infty \in dv) \nu_{1/2}(dz) > 0.$$

Note that in the function φ, z is changed to $-z$ for duality reasons (see [2] for details). As U_∞ and V_∞ are a.s. finite with respect to the corresponding measures, this yields the claim. $\square$

Remark. The proof may be adapted to the lattice case, by proving for example that

$$\liminf_{n \to \infty} \mathbf{P}^+\Big(\sum_{i=0}^{n/2} \eta_{i+1} e^{-S_i} < d, S_n/\sqrt{n} \in (a, b)\Big) > 0.$$

We note that $\mathbf{P}^+(\sum_{i=0}^{\infty} \eta_{i+1} e^{-S_i} < \infty) = \mathbf{P}^-(\sum_{i=1}^{\infty} \eta_i e^{S_i} < \infty) = 1$ has been proved in [4] also for the non-lattice case. But the main remaining problem is that Proposition 2.5 in [2] is only stated for non-lattice random walks. The generalization of this result is a technically involved task and beyond the scope of this paper.

5.2 Proof of Proposition 2.2 (i) : $\rho > 0$

Under Assumption 1 , we now prove that $\rho > 0$. It means that the probability of staying small but alive is exponentially small. The proof relies again on the Geiger construction and results of the previous section.

We assume that there exists $\gamma > 0$ such that $Q(0) < 1-\gamma$ a.s. and $\mathbb{E}[|X|] < \infty$. Let z_0 be the smallest element in $\mathcal{I}$. Using (4.3) and (4.4), we get

$$\begin{aligned}\lim_{n\to\infty} \tfrac{1}{n}\log \mathbb{P}_{z_0}(Z_n = z_0) &= \lim_{n\to\infty} \tfrac{1}{n}\log \mathbb{E}\big[\mathbb{P}_{z_0}(\hat{Z}_n^{(0)}+\ldots+\hat{Z}_n^{(n-1)}=0, \hat{Y}_n = z_0-1|\mathcal{E})\mathbb{P}_{z_0}(Z_n>0|\mathcal{E})\big]\\ &\le \liminf_{n\to\infty} \tfrac{1}{n}\log \mathbb{E}\Big[\prod_{j=0}^{n-1}\mathbb{P}(\hat{Z}_n^{(j)}=0|\mathcal{E})\Big]\\ &= \liminf_{n\to\infty} \tfrac{1}{n}\log \mathbb{E}\Big[\exp\Big(\sum_{j=0}^{n-1}\log \mathbb{P}(\hat{Z}_n^{(j)}=0|\mathcal{E})\Big)\Big].\end{aligned}$$

The fact that $\log(x) \le x-1$ yields

$$\mathbb{E}\Big[\exp\Big(\sum_{j=0}^{n-1}\log \mathbb{P}(\hat{Z}_n^{(j)}=0|\mathcal{E})\Big)\Big] \le \mathbb{E}\Big[\exp\Big(-\sum_{j=0}^{n-1}\mathbb{P}(\hat{Z}_n^{(j)}>0|\mathcal{E})\Big)\Big].$$

It remains to prove that this last expectation decreases exponentially. From (3.5), we get

$$\mathbb{P}(\hat{Z}_n^{(j)}=0|\mathcal{E}) = \frac{p_{j,n}}{p_{j-1,n}}f_j'(f_{j,n}(0)) = \frac{1-f_{j,n}(0)}{1-f_j(f_{j,n}(0))}f_j'(f_{j,n}(0)) = h_j\big(f_{j,n}(0)\big), \tag{5.3}$$

where for $s\in[0,1)$,

$$h(s) := \frac{f'(s)}{g(s)}, \qquad g(s) := \frac{1-f(s)}{1-s}.$$

We will now show that $g(1) = f'(1)$, $h(1) = 1$. As already noticed in [12], for every $s\in[0,1)$

$$g(s) = \sum_{k=0}^{\infty}\frac{1-s^k}{1-s}\frac{f^{(k)}(0)}{k!} = \sum_{k=1}^{\infty}(1+s+s^2+\ldots+s^{k-1})\frac{f^{(k)}(0)}{k!}.$$

Thus $f'(1) = g(1) = 1$ and $h(1) = 1$. Moreover, $f'(0) \neq 1$ ensures that for every $k > 1$ and $s < 1$, $ks^{k-1} < (1+s+s^2+\ldots+s^{k-1})$, so $h(s) < 1$. A straightforward calculation shows that h has exactly one minimum in some $s_0 \in (0,1)$. Adding that h is increasing for $s > s_0$ and $h(s_0) \le h(0)$, we have every $t\in(0,1)$ that

$$h(s) \le \max\big\{h(0), h(t)\big\} \text{ for } s \le t, \tag{5.4}$$

First, we deal with $f_{j,n}(0) = \mathbb{P}(Z_n = 0|\mathcal{E}, Z_j = 1)$ in (5.3). For this purpose, we use a truncation argument. Let $a\in\mathbb{N}$ be fixed for the moment and introduce

$$\bar{Q}(j) := Q(j), 1\le j< a,\ \bar{Q}(a) = Q([a,\infty)).$$

We refrain from indicating the dependence on a in our notation. The corresponding truncated random variables are denoted similarly, e.g. by $\bar{X}$, $\bar{S}$, $\bar{f}$. Note that $\bar{f}''(1) \le a^2$. By dominated convergence, we get that

$$\lim_{a\to\infty}\mathbb{E}[\bar{X}] = \mathbb{E}[X] > 0.$$

Thus if a is chosen large enough, $\bar{S}$ is still a random walk with positive drift and $\mathbb{E}[\bar{f}'(1)] > 1$. Also note that with respect to the truncated offspring distributions, $\bar{Z}_n$ is stochastically smaller than

Z_n and thus $\mathbb{P}(\bar{Z}_n = 0) \geq \mathbb{P}(Z_n = 0)$. Applying this together with a well-known formula for the extinction probability (see e.g. [5][Lemma 2]), we get that

$$\mathbb{P}(Z_n = 0|\mathcal{E}, Z_j = 1) \leq \mathbb{P}_1(\bar{Z}_n = 0|\mathcal{E}) \leq 1 - \frac{1}{e^{-\bar{S}_n} + \sum_{k=j}^{n-1} \bar{\eta}_{k+1} e^{-(\bar{S}_k - \bar{S}_j)}}$$
$$\leq 1 - \frac{1}{a^2 \sum_{k=j}^{\infty} e^{-2\bar{X}_{k+1} - (\bar{S}_k - \bar{S}_j)}},$$

where we used that $\bar{\eta} = \bar{f}''(1)/\bar{f}'(1)^2 \leq a^2 e^{-2\bar{X}}$ a.s. We now aim at bounding $\sum_{k=j}^{\infty} \exp(-2\bar{X}_{k+1} - (\bar{S}_k - \bar{S}_j))$. First, the assumption $Q(0) < 1 - \gamma$ implies that $\bar{X} \geq \log(\gamma)$ a.s. Thus

$$\mathbb{P}(Z_n = 0|\mathcal{E}, Z_j = 1) \leq \mathbb{P}_1(\bar{Z}_n = 0|\mathcal{E}) \leq 1 - \frac{\gamma^2}{a^2 \sum_{k=j}^{\infty} e^{-(\bar{S}_k - \bar{S}_j)}}.$$

Next, we introduce the random walk $\breve{S}_n := \bar{S}_n - \varepsilon n$ with $0 < \varepsilon < \mathbb{E}[\bar{X}]$. It is still a random walk with positive drift and we have

$$f_{j,n}(0) \leq 1 - \frac{\gamma^2}{a^2 \sum_{k=j}^{\infty} e^{-(\breve{S}_k - \breve{S}_j)} e^{-(k-j)\varepsilon}}.$$

Let us now consider the prospective minima (see e.g. [4][p.661]) of $\breve{S}$ which are defined by $\nu(0) := 0$ and

$$\nu(j) := \inf\{n > \nu(j-1) \ : \ \breve{S}_k > \breve{S}_n \ \forall k > n\}.$$

Then we can estimate for $j \geq 1$ (note that $\breve{S}_k \geq \breve{S}_{\nu(j)}$ for all $k \geq \nu(j)$, $j \geq 1$)

$$f_{\nu(j),n}(0) \leq 1 - \frac{\gamma^2}{a^2 \sum_{k=\nu(j)}^{\infty} e^{-(\breve{S}_k - \breve{S}_{\nu}(j))} e^{-(k-\nu(j))\varepsilon}}$$
$$\leq 1 - \frac{\gamma^2}{a^2 \sum_{k=\nu(j)}^{\infty} e^{-(k-\nu(j))\varepsilon}} = 1 - \frac{\gamma^2(1 - e^{-\varepsilon})}{a^2}.$$

From (5.4), setting $d := \gamma^2(1 - e^{-\varepsilon})/a^2 \in (0, 1)$, we get for $j \geq 1$,

$$\mathbb{P}(\hat{Z}_n^{(\nu(j))} > 0|\mathcal{E}) = 1 - h_{\nu(j)}\big(f_{\nu(j),n}(0)\big) \geq 1 - \max\{h(0), h_{\nu(j)}\big(1 - d\big)\}$$
$$= \min\{1 - h(0), 1 - h_{\nu(j)}\big(1 - d\big)\} =: A_{\nu(j)},$$

From the classical random walk theory, $U_j := \nu(j) - \nu(j-1)$ (and also $Q_{\nu(j)}$) are i.i.d. random variables (see [4]). We now prove that for $\delta > 0$ small enough that the probability that there are less than $\{\delta n\}$-many prospective minima is exponentially small. Note that $\mathbb{E}[\breve{X}] > 0$ implies $\mathbb{E}[\nu(1)] < \infty$. Let $0 < \delta < \mathbb{E}[\nu(1)]^{-1}$. Then

$$\mathbb{P}(\sharp\{j \geq 0 \ : \nu(j) \leq n\} < \delta n) \leq \mathbb{P}(\nu(\lceil \delta n \rceil) > n) \leq \mathbb{P}\Big(\sum_{j=1}^{\lfloor \delta n \rfloor} U_j > \frac{1}{\delta} n\delta\Big) \leq e^{-\delta n \Psi(\delta^{-1})}, \tag{5.5}$$

where Ψ is the rate function of the process $(\sum_{j=1}^{n} U_j)_n$, which is a random walk with nonnegative increments. Thus it remains to prove that $\Psi(\theta) > 0$ for some $\theta > \mathbb{E}[U_1] = \mathbb{E}[\nu(1)]$. From large deviations theory, we just need to check that the tail of $\nu(1)$ decreases exponentially. This follows from

$$\mathbb{P}(\nu(1) > k) \leq \mathbb{P}(S_j \leq 0 \ for\ some\ j > k) \leq \sum_{j=k}^{\infty} \mathbb{P}(S_j \leq 0)$$
$$\leq \sum_{j=k}^{\infty} e^{-\breve{\Lambda}(0)j} = \frac{e^{-\breve{\Lambda}(0)k}}{1 - e^{-\breve{\Lambda}(0)}},$$

where $\breve{\Lambda}$ is the rate function of $\breve{S}$. This rate function is proper since $\log(1-\gamma) \leq \breve{X} \leq a$ a.s. Adding that $\mathbb{E}[\breve{X}] > 0$ ensures that $\breve{\Lambda}(0) > 0$ and we can conclude that $\Psi(\theta) > 0$.
Finally, we use that $A_{\nu(j)}$ are independent to get

$$\begin{aligned}\mathbb{P}_{z_0}(Z_n = z_0) &\leq \mathbb{E}\Big[\exp\Big(-\sum_{j=0}^{n-1}\mathbb{P}(\hat{Z}_n^{(j)} > 0|\mathcal{E})\Big)\Big] \\ &\leq \mathbb{P}(\sharp\{j \geq 0\ : \nu(j) \leq n\} < \delta n) + \mathbb{E}\Big[\exp\Big(-\sum_{j=0}^{\lfloor \delta n\rfloor} A_{\nu(j)}\Big)\Big] \\ &\leq \mathbb{P}(\sharp\{j \geq 0\ : \nu(j) \leq n\} < \delta n) + \mathbb{E}\Big[\exp\Big(-A_{\nu(1)}\Big)\Big]^{\lfloor \delta n\rfloor}.\end{aligned}$$

Recalling that $A_{\nu(j)} \geq 0$ and $\mathbb{P}(A_{\nu(j)} > 0) = \mathbb{P}(h(0) < 1) = \mathbb{P}(f'(0) \neq 1) > 0$, we get

$$\mathbb{E}\Big[\exp\Big(-A_{\nu(j)}\Big)\Big] < 1.$$

Then (5.5) ensures that $\rho > 0$. $\square$

Remark. *To get Proposition 2.2 (i), Assumption* 1 *can be replaced by assuming that there exists* c *such that* $\eta \leq c$ *a.s. The proof is very similar. In this case, the truncation is not required and we may estimate*

$$\mathbb{P}(Z_n = 0|\mathcal{E}, Z_j = 1) \leq \bar{\mathbb{P}}(Z_n = 0) \leq 1 - \frac{1}{e^{-S_n} + \sum_{k=j}^{n-1}\eta_{k+1}e^{-(S_k - S_j)}} \leq 1 - \frac{1}{c\sum_{k=j}^{\infty} e^{-(S_k-S_j)}}.$$

5.3 Proof of Proposition 2.2 (ii) : $\varrho \leq \Lambda(0)$

Here, we prove the second part of Proposition 2.2 which ensures that $\varrho \leq \Lambda(0)$. It means that small but positive values can always be realized by a suitable exceptional environment, which is 'critical'. We focus on the nontrivial case when $\Lambda(0) < \infty$. The proof of Proposition 2.2 (ii) can then be splitted into two subcases, which correspond to the two following propositions.

Proposition 5.5. *Under Assumption 2 and* $\mathbb{P}(X < 0) > 0$*, we have* $\rho \leq \Lambda(0)$.

Proposition 5.6. *Assume that* $\mathbb{P}(X \geq 0) = 1$ *and* $\mathbb{P}(X = 0) > 0$*. Then*

$$\rho \leq -\log \mathbb{P}(X = 0) = \Lambda(0). \tag{5.6}$$

Proof of Proposition 5.5. We recall that $\mathcal{I} := \big\{j \geq 1\ :\ \mathbb{P}(Q(j) > 0, Q(0) > 0) > 0\big\}$. We use a standard approximation argument and consider the event $E_{x,n} := \{\min_{i=1,\dots,n} X_i > x\}$ for $x < 0$. Then, $\mathbb{P}(X > x) > 0$ since we are in the supercritical regime and for every $s \geq 0$, $\mathbb{E}\big[|X|e^{-sX}|X > x\big] < \infty$. As $\mathbb{P}(X < 0) > 0$, we may choose x small enough such that $\mathbb{P}(x < X < 0) > 0$. Then $\mathbb{E}\big[|X|e^{-sX}|X > x\big]$ tends to infinity as $s \to \infty$. Moreover $\mathbb{E}[e^{-sX}|X > x]$ is differentiable with respect to s for $s > 0$. We call $s = \nu_x$ a point where the minimum of this function is reached. In particular,

$$\inf_{s\geq 0}\mathbb{E}[e^{-sX}|X > x] = \mathbb{E}[e^{-\nu_x X}|X > -x], \qquad \frac{d}{ds}\mathbb{E}[e^{-sX}|X > x]\big|_{s=\nu_x} = \mathbb{E}[Xe^{-\nu_x X}|X > x] = 0.$$

The second part of Lemma 4.1 ensures that for every $z \in \mathcal{I}$ and for every sequence $k_n = o(n)$,

$$\lim_{n\to\infty}\frac{1}{n}\log \mathbb{P}_z(Z_n = z) = \lim_{n\to\infty}\frac{1}{n}\log \mathbb{P}_z(1 \leq Z_n \leq k_n) = -\rho.$$

Let us now change to the measure $\mathbf{P}$, defined by

$$\mathbf{P}(X \in dy) = \frac{e^{-\nu_x y}\mathbb{P}(X \in dy | X > x)}{\mu} \tag{5.7}$$

where $\mu := \mathbb{E}[e^{-\nu_x X}|X > x]$. Under $\mathbf{P}$, $\mathbf{E}[X] = 0$ and S is a recurrent random walk.
Let $c > 0$ be so large such that $\mathbf{P}(L_n \geq 0, S_n \leq c) > 0$ for every n. Then

$$\begin{aligned}\mathbb{P}_z(1 \leq Z_n \leq k_n | E_{x,n}) &= \mu^n \mathbf{E}\Big[\mathbb{P}_z(1 \leq Z_n \leq k_n|\mathcal{E})e^{\nu_x S_n}\Big] \\ &\geq \mu^n \mathbf{E}\Big[\mathbb{P}_z(1 \leq Z_n \leq k_n|\mathcal{E}); L_n \geq 0, S_n \leq c\Big]. \end{aligned}\tag{5.8}$$

We note that $\mathbb{P}_z(1 \leq Z_n \leq k_n|\mathcal{E}) = \mathbb{P}_z(Z_n > 0|\mathcal{E}) - \mathbb{P}_z(Z_n > k_n|\mathcal{E})$ a.s. and by Markov inequality, $\mathbb{P}_z(Z_n > k|\mathcal{E}) \leq \frac{ze^{S_n}}{k}$ a.s. It ensures that

$$\mathbb{P}_z(1 \leq Z_n \leq k_n|E_{x,n}) \geq \mu^n \mathbf{E}\Big[\mathbf{P}(Z_n > 0|\mathcal{E}) - ze^c/k_n; L_n \geq 0, S_n \leq c\Big].$$

Plugging this into (5.8) and setting $b_n := \mathbf{P}(L_n \geq 0, S_n \leq c)$, we get

$$\begin{aligned}\mathbb{P}_z(1 \leq Z_n \leq k_n) &\geq \mathbb{P}_z(1 \leq Z_n \leq k_n|E_{x,n})\mathbb{P}(E_{x,n}) \\ &= \mu^n b_n\Big[\mathbf{P}(Z_n > 0|L_n \geq 0, S_n \leq c) - ze^c/k_n\Big]\mathbb{P}(X > x)^n. \end{aligned}\tag{5.9}$$

By construction of $\mathbf{E}$, $\mathbf{Var}(X) \leq \mu^{-1}\mathbb{E}[X^2 e^{-\nu_x X}|X > x] < \infty$. Then from Lemma 5.2, we have $b_n = O(n^{-3/2})$ and $\lim_{n\to\infty} \frac{1}{n}\log b_n = 0$. Let $k_n = n^{-1/2}$. The fact that Assumption 2 holds under $\mathbb{P}$ entails that it holds also under $\mathbf{P}$. Indeed,

$$\begin{aligned}\mathbb{E}\big[(\log^+ \xi_Q(a))^{2+\varepsilon}|X > x\big] &= \mu\mathbf{E}\big[e^{\nu_x X}(\log^+ \xi_Q(a))^{2+\varepsilon}\big] \\ &\geq \mu e^{\nu_x x}\mathbf{E}\big[(\log^+ \xi_Q(a))^{2+\varepsilon}\big],\end{aligned}$$

as $X > x$ ($x < 0$) $\mathbf{P}$-a.s. Thus we can use Lemma 5.4 and (5.9) to get

$$\begin{aligned}\liminf_{n\to\infty} \tfrac{1}{n}\log\mathbb{P}_z(1 \leq Z_n \leq k_n) &\geq \log\mu + \log\mathbb{P}(X > x) \\ &= \log\mathbb{E}[e^{-\nu_x X}|X > x] + \log\mathbb{P}(X > x) \\ &= -\sup_{s\leq 0}\big\{-\log\mathbb{E}[e^{-sX}; X > x]\big\}.\end{aligned}$$

By monotone convergence, we let $x \to -\infty$ and

$$\liminf_{n\to\infty} \tfrac{1}{n}\log\mathbb{P}_z(1 \leq Z_n \leq k_n) \geq -\sup_{s\leq 0}\big\{-\log\mathbb{E}[e^{-sX}]\big\} = -\Lambda(0).$$

As $k_n = o(n)$, we apply Lemma 4.1 to end up the proof. □

Proof of Proposition 5.6. As $\mathbb{P}(X \geq 0) = 1$, we have $\mathbb{P}(S_n = 0) = \mathbb{P}(X = 0)^n$ and $\Lambda(0) = -\log\mathbb{P}(X = 0)$.
If $\mathbb{P}(Q(1) = 1|X = 0) = 1$, the proof is trivial. So let us work with the assumptions $\mathbb{P}(X = 0) > 0$ and $\mathbb{P}(Q(1) = 1|X = 0) < 1$.
By conditioning on the environment, we get for $z \in \mathcal{I}$ that

$$\mathbb{P}_z(Z_n = z) \geq \mathbb{P}(X = 0)^n \cdot \mathbb{P}_z(Z_n = z|X_1 = 0, \ldots, X_n = 0).$$

For simplicity, we introduce a new measure $\bar{\mathbf{P}}$ on the space of all probability measures on $\mathbb{N}_0$ with expectation 1. It is defined for every measurable $A \subset \Delta$ by

$$\bar{\mathbf{P}}(Q \in A) := \frac{\mathbb{P}(Q \in A; m_Q = 1)}{\mathbb{P}(m_Q = 1)} = \frac{\mathbb{P}(Q \in A; m_Q = 1)}{\mathbb{P}(X = 0)}.$$

Note that $\bar{\mathbf{P}}(X=0)=1$ and there exists $z\geq 1$ such that $\bar{\mathbf{P}}\big(Q(z)>0, Q(0)>0\big)>0$. With respect to $\bar{\mathbf{P}}$, $(Z_n : n\in\mathbb{N}_0)$ is still a branching process in random environment. By Lemma 4.1, there exists $\bar{\rho}\in[0,\infty)$ such that

$$\begin{aligned}\bar{\rho} &= -\lim_{n\to\infty}\tfrac{1}{n}\log\bar{\mathbf{P}}_z(Z_n=z)\\ &= -\lim_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_z(Z_n=z|X_1=0,\dots,X_n=0)\\ &= -\lim_{n\to\infty}\tfrac{1}{n}\log\mathbb{E}\Big[Q_n(z)f_{0,n}^{z-1}(0)\prod_{i=1}^{n-1}f_i'\big(f_{i,n}(0)\big)\Big|X_1=0,\dots,X_n=0\Big].\end{aligned}$$

Next, we use convexity arguments. First, for all $i\leq k$ and $s\in[0,1]$, $f_{i,k}(s)\geq 1-f_{i,k}'(1)(1-s)$. As $\bar{\mathbf{P}}(f_{i,k}'(1)=1)=1$, we get

$$f_{i,k}(s)\geq s \quad \bar{\mathbf{P}}\text{-a.s.} \tag{5.10}$$

Also recall that $f_{0,n}(0)=f_{0,n-1}(f_n(0))$ and by (5.10), $f_{i,n}(0)\geq f_n(0)=Q_n(0)$. Thus, for every $a\in\mathbb{N}$,

$$\begin{aligned}&\liminf_{n\to\infty}\tfrac{1}{n}\log\mathbb{E}\Big[Q_n(z)f_{0,n}^{z-1}(0)\prod_{i=1}^{n-1}f_i'\big(f_{i,n}(0)\big)\Big|X_1=0,\dots,X_n=0\Big]\\ &\qquad\geq\liminf_{n\to\infty}\tfrac{1}{n}\log\bar{\mathbf{E}}\Big[Q_n(z)Q_n(0)^{z-1}\prod_{i=1}^{n-a}f_i'\big(f_{n-a,n}(0)\big)\prod_{i=n-a+1}^{n-1}f_i'\big(f_{i,n}(0)\big)\Big].\end{aligned}$$

For $\varepsilon>0$ fixed, we choose $k=k_\varepsilon\in\mathbb{N}$ large enough such that $\bar{\mathbf{P}}(Q([1,k])>\varepsilon)\geq 1-\varepsilon$. Then, conditionally on $\{Q([1,k])>\varepsilon\}$, $f'(s)\geq\sum_{j=1}^{k}Q(k)s^k\geq\varepsilon s^k$ a.s. for $s\in[0,1]$. Using this inequality, (5.10) and $f_{i,n}(0)\leq f_{i,n-1}(0)$, we have

$$\begin{aligned}&\bar{\mathbf{E}}\Big[Q_n(z)Q_n(0)^{z-1}\prod_{i=1}^{n-a}f_i'\big(f_{n-a,n}(0)\big)\prod_{i=n-a+1}^{n-1}f_i'\big(f_{i,n}(0)\big)\Big|Q_1,\dots,Q_{n-a}\Big]\\ &\geq\bar{\mathbf{E}}\Big[Q_n(z)Q_n(0)^{z-1}\prod_{i=1}^{n-a}f_i'\big(f_{n-a,n}(0)\big)\\ &\qquad\times\prod_{i=n-a+1}^{n-1}f_i'\big(Q_n(0)\big);Q_{n-a+1}([1,k])>\varepsilon,\dots,Q_{n-1}([1,k])>\varepsilon\Big|Q_1,\dots,Q_{n-a}\Big]\\ &\geq\bar{\mathbf{E}}\Big[Q_n(z)Q_n(0)^{z-1}\prod_{i=1}^{n-a}f_i'\big(f_{n-a,n-1}(0)\big)\\ &\qquad\times\prod_{i=n-a+1}^{n-1}\varepsilon Q_n(0)^k;Q_{n-a+1}([1,k])>\varepsilon,\dots,Q_{n-1}([1,k])>\varepsilon\Big|Q_1,\dots,Q_{n-a}\Big]\\ &\geq\bar{\mathbf{E}}\Big[\prod_{i=1}^{n-a}f_i'\big(f_{n-a,n-1}(0)\big);Q_{n-a+1}([1,k])>\varepsilon,\dots,Q_{n-1}([1,k])>\varepsilon\Big|Q_1,\dots,Q_{n-a}\Big]\\ &\qquad\times\bar{\mathbf{E}}\Big[\varepsilon^{a-2}Q_n(z)Q_n(0)^{z-1+(a-2)k}\Big],\end{aligned}$$

where the second expectation is strictly positive as $\bar{\mathbf{P}}(Q(z)>0,Q(0)>0)>0$. The product of two generating functions (and thus the product of finitely many) is again convex. Indeed generating functions, as well as all their derivatives are convex, nonnegative and nondecreasing functions, thus

$$(fg)''=f''g+2g'f'+fg''\geq 0.$$

Similarly, the product of the derivatives of generating functions is again convex. For more details on the product of nonnegative, convex and nondecreasing functions, we refer to [30]. Applying Jensen's inequality to the convex function $\Pi_{i=1}^{n-a} f_i'$, the independence of the environments ensures that

$$\begin{aligned}
&\bar{\mathbf{E}}\Big[\prod_{i=1}^{n-a} f_i'\big(f_{n-a,n-1}(0)\big); Q_{n-a+1}([1,k]) > \varepsilon, \dots, Q_{n-1}([1,k]) > \varepsilon \Big| Q_1, \dots, Q_{n-a}\Big] \\
&\quad \geq \prod_{i=1}^{n-a} f_i'\big(\bar{\mathbf{E}}\big[f_{n-a,n-1}(0); Q_{n-a+1}([1,k]) > \varepsilon, \dots, Q_{n-1}([1,k]) > \varepsilon \big| Q_1, \dots, Q_{n-a}\big]\big) \\
&\qquad\qquad = \prod_{i=1}^{n-a} f_i'\big(\bar{\mathbf{E}}\big[f_{0,a-1}(0); Q_1([1,k]) > \varepsilon, \dots, Q_{a-1}([1,k]) > \varepsilon\big]\big) \qquad \bar{\mathbf{P}}\text{-a.s.}
\end{aligned}$$

Using this inequality yields

$$\begin{aligned}
&\liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}\Big[f_{0,n}^{z-1}(0) \prod_{i=1}^{n} f_i'\big(f_{i,n}(0)\big) \Big| X_1 = 0, \dots, X_n = 0\Big] \\
&\quad \geq \liminf_{n\to\infty} \tfrac{1}{n} \log \Bigg(\bar{\mathbf{E}}\Big[\prod_{i=1}^{n-a} f_i'\big(\bar{\mathbf{E}}\big[f_{0,a-1}(0); Q_1([1,k]) > \varepsilon, \dots, Q_{a-1}([1,k]) > \varepsilon\big]\big)\Big] \\
&\qquad\qquad\qquad\qquad \times \bar{\mathbf{E}}\Big[\varepsilon^{a-2} Q_n(z) Q_n(0)^{z-1}(0) Q_n(0)^{(a-2)k}\Big]\Bigg) \\
&\quad = \liminf_{n\to\infty} \tfrac{1}{n} \log \bar{\mathbf{E}}\Big[f'\big(\bar{\mathbf{E}}\big[f_{0,a-1}(0); Q_1([1,k]) > \varepsilon, \dots, Q_{a-1}([1,k]) > \varepsilon\big]\big)\Big]^{n-a} \\
&\quad = \log \bar{\mathbf{E}}\Big[f'\big(\bar{\mathbf{E}}\big[f_{0,a-1}(0); Q_1([1,k]) > \varepsilon, \dots, Q_{a-1}([1,k]) > \varepsilon\big]\big)\Big].
\end{aligned}$$

Finally, Z is a critical branching process in random environment under the probability $\bar{\mathbf{P}}$ so $\bar{\mathbf{P}}(Z_{a-1} = 0|\mathcal{E}) = f_{0,a-1}(0) \to 1$ $\bar{\mathbf{P}}$-a.s. as $a \to \infty$ (see e.g. [33]). Letting $a \to \infty$, $\varepsilon \to 0$ and recalling that $\bar{\mathbf{P}}(Q([1,k]) > \varepsilon) \geq 1 - \varepsilon$ yields by dominated convergence

$$\log \bar{\mathbf{E}}\Big[f'\big(\bar{\mathbf{E}}\big[f_{0,a-1}(0); Q_1([1,k]) > \varepsilon, \dots, Q_{a-1}([1,k]) > \varepsilon\big]\big)\Big] \to \log \bar{\mathbf{E}}\big[f'(1)\big] = 0.$$

Then,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}\Big[f_{0,n}^{z-1}(0) \prod_{i=1}^{n-1} f_i'\big(f_{i,n}(0)\big) \Big| X_1 = 0, \dots, X_n = 0\Big] \geq 0.$$

This yields the claim. □

Remark. *Note that the bound $f'(s) \leq f'(1)$ for $s \in [0,1]$ immediately yields that $\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_z(Z_n = z) \leq \log \mathbb{E}[X]$. In particular, we recover that for a BPRE with $X = 0$ a.s., the probability of staying bounded but positive is not exponentially small.*

6 The linear fractional case : Proof of Corollary 2.3

In this section, we assume that the offspring distributions have generating functions of linear fractional form, i.e.

$$f(s) \ = \ 1 - \frac{1-s}{m^{-1} + b\, m^{-2}(1-s)/2},$$

where $m = f'(1)$ and $b = f''(1)$.

Under this assumption, direct calculations with generating functions are feasible, i.e. we can explicitly calculate the generating function of Z_n, conditioned on the environment. We also assume that $\mathbb{E}[|X|] < \infty$, $\mathbb{P}(Z_1 = 0) > 0$ and that either $\mathbb{P}(X \geq 0) = 1$ or Assumption 2 hold, such that Proposition 2.2 (ii) holds.

In the next subsection, we prove Corollary 2.3. It gives an expression of ϱ which depends on the sign of $\mathbb{E}[X \exp(-X)]$. Afterwards, we prove Corollary 2.4 which concerns the MRCA. Let us define $\eta_k := b_k m_k^{-2}/2$ and recall that $f_{j,n} = f_{j+1} \circ \ldots \circ f_n$. Then for all $n \in \mathbb{N}$ and $s \in [0,1]$ (see [28, p. 156])

$$f_{0,n}(s) = 1 - \frac{(1-s)}{e^{-S_n} + (1-s)\sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k}}.$$

resp.

$$f_{j,n}(0) = 1 - \frac{1}{e^{-(S_n - S_j)} + \sum_{k=j+1}^{n} \eta_k e^{-(S_{k-1} - S_j)}}. \tag{6.1}$$

Let us state some direct consequences resulting from this formula which will be used later. Taking the derivative,

$$f'_{0,n}(s) = \frac{e^{-S_n}}{\left(e^{-S_n} + (1-s)\sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k}\right)^2}. \tag{6.2}$$

Note that for every $s \in [0,1)$, applying (6.1)

$$\begin{aligned} f'_{0,n}(s)(1-s)^2 &= \frac{e^{-S_n}}{\left((1-s)^{-1} e^{-S_n} + \sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k}\right)^2} \\ &\leq e^{-S_n} \frac{1}{\left(e^{-S_n} + \sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k}\right)^2} \\ &= e^{-S_n}\left(1 - f_{0,n}(0)\right)^2 = e^{-S_n} \mathbb{P}(Z_n > 0 | \mathcal{E})^2. \end{aligned} \tag{6.3}$$

Moreover,

$$f'_j(s) = \frac{e^{-X_j}}{(e^{-X_j} + \eta_j(1-s))^2} \tag{6.4}$$

and we can now compute the value of ϱ.

6.1 Determination of the value of ϱ

By Proposition 2.2 (ii), $\rho \leq \Lambda(0)$. Then it remains to prove that $\rho = -\log \mathbb{E}\left[e^{-X}\right]$ if $\mathbb{E}[X e^{-X}] \geq 0$ and $\rho \geq \Lambda(0)$ otherwise. For that purpose, we use the representation of ρ in terms of generating functions. Combining (6.1) and (6.4) we get

$$\begin{aligned} f'_j\left(f_{j,n}(0)\right) &= e^{-X_j}\left(e^{-X_j} + \frac{\eta_j}{e^{-(S_n - S_j)} + \sum_{k=j+1}^{n} \eta_k e^{-(S_{k-1} - S_j)}}\right)^{-2} \\ &= e^{-X_j}\left(\frac{e^{-(S_n - S_j)} + \sum_{k=j+1}^{n} \eta_k e^{-(S_{k-1} - S_j)}}{e^{-(S_n - S_{j-1})} + \sum_{k=j}^{n} \eta_k e^{-(S_{k-1} - S_{j-1})}}\right)^2 \\ &= e^{-X_j}\left(\frac{\mathbb{P}(Z_n > 0 | Z_{j-1} = 1, \mathcal{E})}{\mathbb{P}(Z_n > 0 | Z_j = 1, \mathcal{E})}\right)^2. \end{aligned}$$

Since $\mathbb{P}(Z_n > 0|Z_n = 1) = 1$, we get

$$\begin{aligned}
\mathbb{P}_1(Z_n = 1) &= \mathbb{E}_1\Big[\prod_{j=1}^{n} f_j'\big(f_{j,n}(0)\big)\Big] \\
&= \mathbb{E}_1\Big[\prod_{j=1}^{n} e^{-X_j}\frac{\mathbb{P}(Z_n > 0|Z_{j-1} = 1, \mathcal{E})^2}{\mathbb{P}(Z_n > 0|Z_j = 1, \mathcal{E})^2}\Big] \\
&= e^{-S_n}\mathbb{P}(Z_n > 0|\mathcal{E}, Z_0 = 1)^2. \qquad (6.5)
\end{aligned}$$

First, we consider the case $\mathbb{E}[Xe^{-X}] \geq 0$. Bounding the probability above by 1 immediately yields

$$\mathbb{E}\Big[\prod_{j=1}^{n} f_j'\big(f_{j,n}(0)\big)\Big] \leq \mathbb{E}[e^{-S_n}] = \mathbb{E}[e^{-X}]^n.$$

Thus $\rho \geq -\log \mathbb{E}[e^{-X}]$. To get the converse inequality, we change to the measure $\hat{\mathbb{P}}$, defined by

$$\hat{\mathbb{P}}(X \in dx) = \frac{e^{-x}\mathbb{P}(X \in dx)}{\mathbb{E}[e^{-X}]}.$$

This measure is well-defined as $\mathbb{E}\big[X^2e^{-X}\big] < \infty$ implies $\mathbb{E}\big[e^{-X}\big] < \infty$. Then by Jensen's inequality,

$$\mathbb{E}_1\Big[e^{-S_n}\mathbb{P}(Z_n > 0|\mathcal{E})^2\Big] = \mathbb{E}\big[e^{-X}\big]^n\hat{\mathbb{E}}\big[\mathbb{P}_1(Z_n > 0|\mathcal{E})^2\big] \geq \mathbb{E}[e^{-X}]^n\hat{\mathbb{P}}_1(Z_n > 0)^2.$$

We observe that $\hat{\mathbb{E}}[X] = \mathbb{E}[Xe^{-X}] \geq 0$, such that under $\hat{\mathbb{P}}$, S_n is a random walk with nonnegative drift. It ensures that the branching process is still critical or supercritical with respect to $\hat{\mathbb{P}}$. Thus, under Assumption 2 and as $\hat{\mathbb{E}}[X] = \mathbb{E}[X^2e^{-X}] < \infty$, $\hat{\mathbb{P}}(Z_n > 0) > Cn^{-\frac{1}{2}}$ for some $C > 0$ as $n \to \infty$ (see e.g. [4] for the critical case, whereas $\mathbb{P}(Z_n > 0)$ has a positive limit in the supercritical case). Letting $n \to \infty$ and adding that $1 \in \mathcal{I}$, we get

$$\rho = -\lim_{n\to\infty} \tfrac{1}{n}\log \mathbb{E}\Big[\prod_{j=1}^{n} f_j'\big(f_{j,n}(0)\big)\Big] \leq -\log \mathbb{E}[e^{-X}].$$

Secondly, we consider $\mathbb{E}[Xe^{-X}] < 0$. Then there exists $\nu \in (0,1)$ such that $\mathbb{E}[Xe^{-\nu X}] = 0$ and we change to the measure $\mathbf{P}$ defined in (5.7) (here without truncation). Applying this change of measure and the well-known estimate $\mathbb{P}(Z_n > 0|\mathcal{E}) \leq e^{L_n\wedge 0}$ a.s., we get that

$$\mathbb{E}\Big[e^{-S_n}\mathbb{P}(Z_n > 0|\mathcal{E}, Z_0 = 1)^2\Big] \leq \mathbb{E}[e^{-\nu X}]^n\mathbf{E}\Big[e^{(-1+\nu)S_n + 2L_n\wedge 0}\Big].$$

Note that $L_n \wedge 0 \leq \min(S_n, 0)$ and $\nu \in (0,1)$ that $(-1+\nu)S_n + 2L_n \wedge 0 \leq 0$, and thus

$$\mathbb{E}\Big[e^{-S_n}\mathbb{P}(Z_n > 0|\mathcal{E}, Z_0 = 1)^2\Big] \leq \mathbb{E}[e^{-\nu X}]^n.$$

This yields $\rho \geq -\log \mathbb{E}[e^{-\nu X}] = \Lambda(0)$ since $\Lambda(0) = \sup_{s\leq 0}\{-\log \mathbb{E}[e^{sX}]\}$ and the condition $\mathbb{E}[Xe^{-\nu X}] = 0$ implies that the supremum is taken in $s = -\nu$.

6.2 Proof of the limit theorems for the MRCA

The three cases (i-ii-iii) in Corollary 2.4 result respectively from Lemma 6.1, Lemmas 6.2, 6.4, 6.5 and Lemma 6.7 below. In all this Section, we assume that the assumptions of Corollary 2.3 are met, i.e. that $\mathbb{E}[X^2e^{-X}] < \infty$, $\mathbb{E}[|X|] < \infty$ and either $\mathbb{P}(X \geq 0) = 1$ or Assumption 2 holds.

Lemma 6.1. *If* $\mathbb{E}[Xe^{-X}] < 0$*, then*

$$\liminf_{n\to\infty} \mathbb{P}_1(MRCA_n = n|Z_n = 2) \in (0,1)$$

and

$$\liminf_{n\to\infty} \mathbb{P}_1(MRCA_n = 1|Z_n = 2) \in (0,1).$$

Proof. Conditionally on $\mathcal{E}$, the branching property holds and ensures that

$$\mathbb{P}_1(MRCA_n = n|Z_n = 2) \geq \mathbb{P}_1(Z_1 = 2) \cdot \frac{\mathbb{E}\big[\mathbb{P}_1(Z_{n-1} = 1|\mathcal{E})^2\big]}{\mathbb{P}_1(Z_n = 2)},$$

which corresponds to two subtrees being founded by $Z_0 = 1$ and staying equal to 1 in all generations. Next, we recall that linear fractional offspring distributions are stable with respect to compositions and have geometrically decaying probability weights (see e.g. [28]). In particular,

$$\mathbb{P}_1(Z_n = 2|\mathcal{E}) \leq \mathbb{P}_1(Z_n = 1|\mathcal{E}) \tag{6.6}$$

and therefore

$$\mathbb{P}_1(MRCA_n = n|Z_n = 2) \geq \mathbb{P}_1(Z_1 = 2) \cdot \frac{\mathbb{E}\big[\mathbb{P}_1(Z_{n-1} = 1|\mathcal{E})^2\big]}{\mathbb{P}_1(Z_n = 1)}. \tag{6.7}$$

Note that $\mathbb{E}[Xe^{-X}] < 0$ implies that there exists $\nu \in (0,1)$ such that $\mathbb{E}[Xe^{-\nu X}] = 0$. Thus we can apply the same change of measure as in the proof of Proposition 5.5. With the definition of $\mathbf{P}$ therein (again without truncation), we get

$$\frac{\mathbb{E}[\mathbb{P}_1(Z_n = 1|\mathcal{E})^2]}{\mathbb{P}_1(Z_n = 1)} = \frac{\mathbf{E}[e^{\nu S_n}\mathbb{P}_1(Z_n = 1|\mathcal{E})^2]}{\mathbf{E}[e^{\nu S_n}\mathbb{P}_1(Z_n = 1|\mathcal{E})]}. \tag{6.8}$$

From (6.5), we know that

$$\mathbb{P}_1(Z_n = 1|\mathcal{E}) = f'_{0,n}(1) = e^{-S_n}\mathbb{P}_1(Z_n > 0|\mathcal{E})^2 \leq e^{-S_n + 2L_n} \text{ a.s.} \tag{6.9}$$

Combining this with Jensen inequality yields for every $c > 0$,

$$\begin{aligned}
\mathbf{E}[e^{\nu S_n}\mathbb{P}_1(Z_n = 1|\mathcal{E})^2] &\geq \mathbf{E}\big[e^{-2S_n + \nu S_n}\mathbb{P}_1(Z_n > 0|\mathcal{E})^4; L_n \geq 0, S_n < c\big] \\
&\geq e^{(-2+\nu)c}\mathbf{E}\big[\mathbb{P}_1(Z_n > 0|\mathcal{E})^4; L_n \geq 0, S_n < c\big] \\
&\geq e^{(-2+\nu)c}\mathbf{P}(L_n \geq 0, S_n < c)\mathbf{E}\big[\mathbb{P}_1(Z_n > 0|\mathcal{E})^4|L_n \geq 0, S_n < c\big] \\
&\geq e^{(-2+\nu)c}\mathbf{P}(L_n \geq 0, S_n < c)\mathbf{P}(Z_n > 0|L_n \geq 0, S_n < c)^4 \\
&\geq d\ \mathbf{P}(L_n \geq 0, S_n < c),
\end{aligned} \tag{6.10}$$

for some constant $d > 0$, where the last line follows from Lemma 5.4. For the denominator in (6.8), by (6.9), we get similarly

$$\mathbf{E}[e^{\nu S_n}\mathbb{P}_1(Z_n = 1|\mathcal{E})] \leq \mathbf{E}[e^{-(1-\nu)S_n + 2L_n}].$$

Finally, we use $\mathbb{E}[X^2\exp(-X)]$ to ensure that $\mathbf{E}[X^2] < \infty$ and apply Lemmas 5.1 and 5.2. Then,

$$\liminf_{n\to\infty} \frac{\mathbf{E}[e^{\nu S_n}\mathbb{P}_1(Z_n = 1|\mathcal{E})^2]}{\mathbf{E}[e^{\nu S_n}\mathbb{P}_1(Z_n = 1|\mathcal{E})]} \geq \liminf_{n\to\infty} \frac{\mathbf{P}(L_n \geq 0, S_n < c)}{\mathbf{E}[e^{-(1-\nu)S_n + 2L_n}]} > 0.$$

Recalling (6.7) yields the first part of the lemma.
Similarly, we note that by the Markov property,

$$\mathbb{P}_1(MRCA_n = 1, Z_n = 2) \geq \mathbb{P}_1(Z_{n-1} = 1)\mathbb{P}(Z_1 = 2).$$

Recalling that $\mathbb{P}_1(Z_{n-1} = 1) \geq \mathbb{P}_1(Z_{n-1} = 2)$ from (6.6) yields the second claim. □

The next three lemmas cover the 'intermediate' regime. First, we prove that the probability of $\{MRCA_n = x_n\}$ doesn't decay exponentially for every $1 \le x_n \le n$.

Lemma 6.2. *If* $\mathbb{E}[Xe^{-X}] = 0$*, then for every* $1 \le x_n \le n$

$$\lim_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_1(MRCA_n = x_n | Z_n = 2) = 0.$$

Proof. Let $\mathbb{E}[Xe^{-X}] = 0$ and note that $MRCA_n \ge 1$. Conditionally on $\mathcal{E}$, the branching property holds and guarantees that as in the previous proof that for every $1 \le x_n \le n$,

$$\mathbb{P}_1(MRCA_n = x_n | Z_n = 2) \ge \mathbb{P}_1(Z_{n-x_n} = 2) \cdot \frac{\mathbb{E}\big[\mathbb{P}_1(Z_{x_n} = 1|\mathcal{E})^2\big]}{\mathbb{P}_1(Z_n = 2)}.$$

As $\mathbb{E}[X\exp(-X)] \ge 0$, we know from the previous subsection that $\log \mathbb{E}[e^{-X}] = \lim_{n\to\infty} \mathbb{P}_1(Z_n = 2)$. Thus we get for $1 \le x_n \le n$,

$$\begin{aligned}
&\liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_1(MRCA_n = x_n | Z_n = 2) \\
&\quad \ge \liminf_{n\to\infty} \Big\{ (1 - x_n/n) \log \mathbb{E}[e^{-X}] + \tfrac{1}{n} \log \mathbb{E}\big[\mathbb{P}_1(Z_{x_n} = 1|\mathcal{E})^2\big] \Big\} - \log \mathbb{E}[e^{-X}].
\end{aligned}$$

For the last term, we use the change of measure of the previous lemma with $\nu = 1$, i.e.

$$\mathbf{P}(X \in dx) = \frac{e^{-x}\mathbb{P}(X \in dx)}{\mathbb{E}[e^{-X}]}.$$

Applying (6.10), we get that

$$\mathbb{E}\big[\mathbb{P}_1(Z_{x_n} = 1|\mathcal{E})^2\big] \ge d\ \mathbb{E}[e^{-X}]^{x_n} \mathbf{P}(L_{x_n} \ge 0, S_{x_n} < c).$$

As $\mathbb{E}[Xe^{-X}] = 0$, S is a recurrent random walk under $\mathbf{P}$. Using again Lemma 5.2, $\mathbf{P}(L_n \ge 0, S_n < c) \sim dn^{-\frac{3}{2}}$, and

$$\begin{aligned}
&\liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_1(MRCA_n = x_n | Z_n = 2) \\
&\quad \ge \log \mathbb{E}[e^{-X}] - \log \mathbb{E}[e^{-X}] + \liminf_{n\to\infty} \Big\{ - x_n/n \log \mathbb{E}[e^{-X}] + x_n/n \log \mathbb{E}[e^{-X}] \Big\} = 0.
\end{aligned}$$

It gives the expected lower bound, whereas the upper bound follows is simply due to the fact that the probabilities are less than 1. □

The following lemma is required to prove limit results for the MRCA in the intermediately supercritical case. To avoid technicalities, we impose an additional moment condition.

Lemma 6.3. *Let* $\mathbb{E}\big[Xe^{-X}\big] = 0$ *and* $\gamma = \mathbb{E}[e^{-X}]$*. Assume that* $\mathbb{E}[f''(1)e^{-X}] < \infty$*. Then*

$$0 < \liminf_{n\to\infty} \gamma^{-n} n^{1/2}\ \mathbb{P}_1(Z_n = 2) \le \limsup_{n\to\infty} \gamma^n n^{1/2}\ \mathbb{P}_1(Z_n = 2) < \infty,$$

i.e. $\mathbb{P}_1(Z_n = 2)$ *is of the order* $\gamma^n n^{-1/2}$.

Proof. Using the change of measure to $\mathbf{P}$ and (6.9),

$$\begin{aligned}
\mathbb{P}_1(Z_n = 2) &= \mathbb{E}\big[e^{-S_n}\mathbb{P}_1(Z_n > 0|\mathcal{E})^2\big] \\
&= \gamma^n \mathbf{E}[\mathbb{P}(Z_n > 0|\mathcal{E})^2] \ge \gamma^n \mathbf{P}(Z_n > 0).
\end{aligned} \tag{6.11}$$

Under $\mathbf{P}$, $\mathbf{E}[X] = 0$ and Z is a critical branching process in random environment. Under our assumptions, there is a constant $d > 0$ such that (see [3][Theorem 1.1])

$$\mathbf{P}(Z_n > 0) \sim d\, n^{-\frac{1}{2}}. \tag{6.12}$$

Following the proof of the upper bound, using (6.11) and that $\mathbb{P}(Z_n > 0) \le e^{L_n}$ a.s., we have

$$\mathbb{P}_1(Z_n = 2) = \gamma^n \mathbf{E}[\mathbb{P}(Z_n > 0|\mathcal{E})^2] \le \gamma^n \mathbf{E}[e^{2L_n}].$$

Our assumptions imply $\mathbf{E}[X^2] < \infty$ and thus, as a direct consequence of [4][Lemma 2.1 and $\int_{-\infty}^0 e^x u(-x)dx < \infty$],

$$\mathbf{E}[e^{2L_n}] = O(n^{-1/2}).$$

This proves the upper bound. □

The next lemma describes the probability of $\{MRCA_n = n\}$ in the intermediate regime.

Lemma 6.4. *Let $\mathbb{E}\big[Xe^{-X}\big] = 0$ and assume that $\mathbb{E}\big[f''(1)\big] < \infty$. Then*

$$0 < \liminf_{n\to\infty} n\ \mathbb{P}_1(MRCA_n = n|Z_n = 2) \le \limsup_{n\to\infty} n\ \mathbb{P}_1(MRCA_n = n|Z_n = 2) < \infty,$$

i.e. $\mathbb{P}_1(MRCA_n = n|Z_n = 2)$ is of the order n^{-1}.

Proof. First, the event $\{MRCA_n = n\}$ implies that there are at least two individuals in generation 1 and that from this generation on, at least two subtrees survive until generation n. We use the branching property and a decomposition according to the two subtrees which survive and get that

$$\begin{aligned}
&\mathbb{P}_1(MRCA_n = n, Z_n = 2|\mathcal{E})\\
&\quad= \sum_{k=2}^{\infty} \binom{k}{2} \mathbb{P}_1(Z_1 = k|\mathcal{E})\mathbb{P}(Z_n = 1|\mathcal{E}, Z_1 = 1)^2 \mathbb{P}(Z_n = 0|\mathcal{E}, Z_1 = 1)^{k-2}\\
&\quad\le \sum_{k=2}^{\infty} \frac{k(k-1)}{2} \mathbb{P}_1(Z_1 = k|\mathcal{E})\mathbb{P}(Z_n = 1|\mathcal{E}, Z_1 = 1)^2\\
&\quad\le \mathbb{P}(Z_n = 1|\mathcal{E}, Z_1 = 1)^2 f_1''(1) \text{ a.s.,}
\end{aligned} \tag{6.13}$$

since $f_1''(1) = \sum_{k=2}^{\infty} k(k-1)\mathbb{P}_1(Z_1 = k|\mathcal{E})$. Using (6.9) and independence yields

$$\begin{aligned}
&\mathbb{P}_1(MRCA_n = n, Z_n = 2)\\
&\quad\le \mathbb{E}[f''(1)]\mathbb{E}[e^{-2S_{n-1}}\mathbb{P}(Z_{n-1} > 0|\mathcal{E})^4] \le \mathbb{E}[f''(1)]\mathbb{E}[e^{-2S_{n-1}+4L_{n-1}}].
\end{aligned} \tag{6.14}$$

Again, we change to the measure $\mathbf{P}$. Note that the assumptions of the lemma $\mathbb{E}[f''(1)] < \infty$ ensures that $\mathbf{Var}(X) < \infty$. Using also Corollary 5.3, s we get

$$\mathbb{P}_1(MRCA_n = n, Z_n = 2) \le \mathbb{E}[f''(1)]\gamma^{n-1}\mathbf{E}[e^{-S_{n-1}+4L_{n-1}}] = O\big(\gamma^n n^{-3/2}\big).$$

Inserting this and applying Lemma 6.3 yields

$$\limsup_{n\to\infty} n\ \mathbb{P}_1(MRCA_n = n|Z_n = 2) < \infty.$$

For the lower bound, we use similar arguments. First,

$$\mathbb{P}_1(MRCA_n = n, Z_n = 2|\mathcal{E}) \ge \mathbb{P}_1(Z_1 = 2|\mathcal{E})\mathbb{P}_1(Z_{n-1} = 1|\mathcal{E}, Z_1 = 1)^2 \quad \text{a.s.}$$

Let $c > 0$. Taking the expectation and using (6.9) yields

$$\begin{aligned}
&\mathbb{P}_1(MRCA_n = n, Z_n = 2) \\
&\quad \geq \mathbb{P}_1(Z_1 = 2)\mathbb{E}[e^{-2S_{n-1}}\mathbb{P}_1(Z_{n-1} > 0|\mathcal{E})^4] \\
&\quad \geq \mathbb{P}_1(Z_1 = 2)\gamma^{n-1}e^{-c}\mathbf{E}\big[\mathbb{P}_1(Z_{n-1} > 0\big|\mathcal{E})^4|L_{n-1} \geq 0, S_{n-1} \leq c\big]\mathbf{P}(L_{n-1} \geq 0, S_{n-1} \leq c).
\end{aligned}$$

Moreover, by Lemma 5.4 and Jensen's inequality,

$$\liminf_{n\to\infty} \mathbf{E}[\mathbb{P}_1(Z_n > 0|\mathcal{E})^4|L_n \geq 0, S_n \leq c] > 0.$$

Applying Lemma 6.3 again, we get that

$$\liminf_{n\to\infty} n\ \mathbb{P}_1(MRCA_n = n|Z_n = 2) > 0.$$

□

The next Lemma describes the probability that the MRCA is neither at the beginning nor at the end:

Lemma 6.5. *Let $\mathbb{E}\big[Xe^{-X}\big] = 0$ and $\mathbb{E}[f''(1)/(1-f(0))^2] < \infty$. Then for every $\delta \in (0,1)$*

$$0 < \liminf_{n\to\infty} n^{3/2}\ \mathbb{P}_1(MRCA_n = \lceil \delta n \rceil|Z_n = 2) \leq \limsup_{n\to\infty} n^{3/2}\ \mathbb{P}_1(MRCA_n = \lceil \delta n \rceil|Z_n = 2) < \infty,$$

i.e. $\mathbb{P}_1(MRCA_n = \lceil \delta n \rceil|Z_n = 2)$ is of the order $n^{-3/2}$.

Proof. First, the event $\{MRCA_n = \lceil \delta n \rceil, Z_n = 2\}$ implies that the two individuals in generation n stem from one individual in generation $n - \lceil \delta n \rceil = \lfloor (1-\delta)n \rfloor$. If there are k individuals in this generation, there are k possibilities for the ancestor from which the two surviving individuals in generation n stem from. All others have to become extinct. We use the branching property and a decomposition according to the two subtrees which survive to get that a.s.

$$\begin{aligned}
&\mathbb{P}(MRCA_n = \lceil \delta n \rceil, Z_n = 2|\mathcal{E}) \\
&\quad = \sum_{k=1}^{\infty} \mathbb{P}(Z_{\lfloor (1-\delta)n \rfloor} = k|\mathcal{E})k\mathbb{P}(Z_n = 0|\mathcal{E}, Z_{\lfloor (1-\delta)n \rfloor} = k-1) \\
&\qquad\qquad \mathbb{P}(MRCA_n = \lceil \delta n \rceil, Z_n = 2|\mathcal{E}, Z_{\lfloor (1-\delta)n \rfloor} = 1) \\
&\quad = \sum_{k=1}^{\infty} \mathbb{P}(Z_{\lfloor (1-\delta)n \rfloor} = k|\mathcal{E})k\mathbb{P}(Z_n = 0|\mathcal{E}, Z_{\lfloor (1-\delta)n \rfloor} = 1)^{k-1} \\
&\qquad\qquad \mathbb{P}(MRCA_n = \lceil \delta n \rceil, Z_n = 2|\mathcal{E}, Z_{\lfloor (1-\delta)n \rfloor} = 1)
\end{aligned}$$

Next, we set

$$s := (s(\mathcal{E}) = \mathbb{P}(Z_n = 0|\mathcal{E}, Z_{\lfloor (1-\delta)n \rfloor + 1} = 1)$$

and note that

$$\mathbb{P}(Z_n = 0|\mathcal{E}, Z_{\lfloor (1-\delta)n \rfloor} = 1) = f_{\lfloor (1-\delta)n \rfloor}(s).$$

Thus we get that a.s.

$$\begin{aligned}
&\mathbb{P}(MRCA_n = \lceil \delta n \rceil, Z_n = 2|\mathcal{E}) \\
&\quad = f'_{0,\lfloor (1-\delta)n \rfloor}(f_{\lfloor (1-\delta)n \rfloor}(s))\mathbb{P}_1(MRCA_n = \lceil \delta n \rceil, Z_n = 2|\mathcal{E}, Z_{\lfloor (1-\delta)n \rfloor} = 1) \qquad (6.15)
\end{aligned}$$

Next, using (6.13) and (6.9), we get that

$$\begin{aligned}
&\mathbb{P}_1(MRCA_n = \lceil \delta n \rceil, Z_n = 2|\mathcal{E}, Z_{\lfloor (1-\delta)n \rfloor} = 1) && (6.16)\\
&\quad \leq f''_{\lfloor (1-\delta)n \rfloor+1}(1)\mathbb{P}(Z_n = 1|\mathcal{E}, Z_{\lfloor (1-\delta)n \rfloor+1} = 1)^2 \\
&\quad = f''_{\lfloor (1-\delta)n \rfloor+1}(1)e^{-2(S_n - S_{\lfloor (1-\delta)n \rfloor+1})}\mathbb{P}(Z_n > 0|\mathcal{E}, Z_{\lfloor (1-\delta)n \rfloor+1} = 1)^4 \\
&\quad = f''_{\lfloor (1-\delta)n \rfloor+1}(1)e^{-2(S_n - S_{\lfloor (1-\delta)n \rfloor+1})}(1-s)^4. && (6.17)
\end{aligned}$$

Combining (6.15) and (6.17), we have a.s.

$$\mathbb{P}(MRCA_n = \lceil \delta n \rceil, Z_n = 2|\mathcal{E}) = f'_{0,\lfloor (1-\delta)n \rfloor}(f_{\lfloor (1-\delta)n \rfloor}(s))f''_{\lfloor (1-\delta)n \rfloor+1}(1)e^{-2(S_n - S_{\lfloor (1-\delta)n \rfloor+1})}(1-s)^4.$$

Moreover, from (6.3),

$$f'_{0,\lfloor (1-\delta)n \rfloor}(f_{\lfloor (1-\delta)n \rfloor}(s))\big(1 - f_{\lfloor (1-\delta)n \rfloor}(s)\big)^2 \leq e^{-S_{\lfloor (1-\delta)n \rfloor}}\mathbb{P}(Z_{\lfloor (1-\delta)n \rfloor} > 0|\mathcal{E})^2 \text{ a.s.}$$

So

$$\begin{aligned}
&\mathbb{P}(MRCA_n = \lceil \delta n \rceil, Z_n = 2|\mathcal{E}) \\
&= e^{-S_{\lfloor (1-\delta)n \rfloor}}\mathbb{P}(Z_{\lfloor (1-\delta)n \rfloor} > 0|\mathcal{E})^2 f''_{\lfloor (1-\delta)n \rfloor+1}(1)e^{-2(S_n - S_{\lfloor (1-\delta)n \rfloor+1})}\frac{(1-s)^4}{(1 - f_{\lfloor (1-\delta)n \rfloor}(s))^2}.
\end{aligned}$$

As already used in [12][Proof of Lemma 1], we have for a generating function f of a random variable R

$$\frac{1-f(s)}{1-s} = \sum_{k=0}^{\infty} s^k \mathbb{P}(R > k),$$

which is obviously increasing in s. Thus we get

$$\frac{1-s}{1 - f_{\lfloor (1-\delta)n \rfloor}(s)} \leq \big(\mathbb{P}(Z_{\lfloor (1-\delta)n \rfloor} > 0|Z_{\lfloor (1-\delta)n \rfloor-1} = 1, \mathcal{E})\big)^{-1} = \frac{1}{1 - f_{\lfloor (1-\delta)n \rfloor}(0)}.$$

Combining these identities and using the independence of the environments yields :

$$\begin{aligned}
&\mathbb{P}(MRCA_n = \lceil \delta n \rceil, Z_n = 2) \\
&\leq \mathbb{E}\big[e^{-S_{\lfloor (1-\delta)n \rfloor}}\mathbb{P}(Z_{\lfloor (1-\delta)n \rfloor} > 0|\mathcal{E})^2\big]\mathbb{E}\big[f''(1)/(1-f(0))^2\big]\mathbb{E}\big[e^{-2S_{\lceil \delta n \rceil-1}}\mathbb{P}_1(Z_{\lceil \delta n \rceil-1} > 0|\mathcal{E})^2\big],
\end{aligned}$$

where we recall that by assumption $\mathbb{E}\big[f''(1)/(1-f(0))^2\big] < \infty$. As we have proved before, for every $\delta \in (0,1)$,

$$\mathbb{E}\big[e^{-S_{\lfloor (1-\delta)n \rfloor}}\mathbb{P}(Z_{\lfloor (1-\delta)n \rfloor} > 0|\mathcal{E})^2\big] \leq \gamma^{\lfloor (1-\delta)n \rfloor}\mathbf{E}\big[e^{2L_{\lfloor (1-\delta)n \rfloor}}\big] = \gamma^{\lfloor (1-\delta)n \rfloor}O(n^{-1/2})$$

and

$$\mathbb{E}\big[e^{-2S_{\lceil \delta n \rceil-1}}\mathbb{P}_1(Z_{\lceil \delta n \rceil-1} > 0|\mathcal{E})^2\big] \leq \gamma^{\lceil \delta n \rceil-1}\mathbf{E}\big[e^{-S_{\lceil \delta n \rceil-1}+2L_{\lceil \delta n \rceil-1}}\big] = \gamma^{\lceil \delta n \rceil-1}O(n^{-3/2}).$$

Together with Lemma 6.3, this yields the expected upper bound:

$$\limsup_{n\to\infty} n^{3/2}\ \mathbb{P}_1(MRCA_n = \lceil \delta n \rceil | Z_n = 2) < \infty.$$

For the lower bound, we use

$$\begin{aligned}
&\mathbb{P}_1(MRCA_n = \lfloor \delta n \rfloor, Z_n = 2|\mathcal{E}) \\
&\quad \geq \mathbb{P}_1(Z_{\lfloor (1-\delta)n \rfloor} = 1|\mathcal{E})\mathbb{P}_1(MRCA_n = \lceil \delta n \rceil, Z_n = 2|\mathcal{E}, Z_{\lfloor (1-\delta)n \rfloor} = 1) \text{ a.s.}
\end{aligned}$$

Both terms are independent and from Lemma 6.4, we get

$$\liminf_{n\to\infty} \gamma^{-\lfloor \delta n\rfloor+1} n^{3/2} \mathbb{P}_1(MRCA_n = \lceil \delta n\rceil, Z_n = 2 | Z_{\lfloor (1-\delta)n\rfloor} = 1) > 0.$$

From the previous lemmas,

$$\liminf_{n\to\infty} \gamma^{-\lfloor (1-\delta)n\rfloor} n^{1/2} \mathbb{P}_1(Z_{\lfloor (1-\delta)n\rfloor} = 1) > 0.$$

Thanks to Lemma 6.3, we obtain the expected lower bound. $\square$

For the next proof, we require the following auxiliary result.

Lemma 6.6. *We assume that* $\mathbb{E}[Xe^{-X}] > 0$*. Then for every* $c > 0$*,*

$$\lim_{n\to\infty} \tfrac{1}{n} \log \inf_{z\geq cn^2} \mathbb{P}_z(Z_n = 2) = -\infty, \qquad \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_1(1 \leq Z_n \leq cn^2) = \log \mathbb{E}[e^{-X}].$$

Proof. First, we observe that on the event $\{Z_n = 2\}$, at most two initial subtrees survive until generation n. Using the the branching property conditionally on $\mathcal{E}$, we have a.s.

$$\begin{aligned}\inf_{z\geq cn^2} \mathbb{P}_z(Z_n = 2|\mathcal{E}) \leq & \sum_{k=\lfloor cn^2\rfloor}^{\infty} \binom{k}{2} \mathbb{P}_1(Z_n = 0|\mathcal{E})^{\lfloor cn^2\rfloor -2} \mathbb{P}_1(1 \leq Z_n \leq 2|\mathcal{E})^2 \\ & + \sum_{k=\lfloor cn^2\rfloor}^{\infty} \binom{k}{1} \mathbb{P}_1(Z_n = 0|\mathcal{E})^{\lfloor cn^2\rfloor -1} \mathbb{P}_1(1 \leq Z_n \leq 2|\mathcal{E}).\end{aligned}$$

Again, we use the geometric form of LF distributions to see that $\mathbb{P}(1 \leq Z_n \leq 2|\mathcal{E}) \leq 2\mathbb{P}(Z_n = 1|\mathcal{E})$ a.s. Next, we use

$$\sum_{k=n}^{\infty} k(k-1)\alpha^{k-2} \leq n^2 \frac{\alpha^{n-2}}{(1-\alpha)^3}, \quad \sum_{k=n}^{\infty} k\ \alpha^{k-1} \leq n \frac{\alpha^{n-1}}{(1-\alpha)^2}$$

to get

$$\begin{aligned}&\inf_{z\geq cn^2} \mathbb{P}_z(Z_n = 2|\mathcal{E}) \\ &\leq n^2 \left[c^2 \frac{\mathbb{P}_1(Z_n = 0|\mathcal{E})^{\lfloor cn^2\rfloor -2} \mathbb{P}_1(Z_n = 1|\mathcal{E})^2}{(1 - \mathbb{P}_1(Z_n = 0|\mathcal{E}))^3} + c \frac{\mathbb{P}_1(Z_n = 0|\mathcal{E})^{\lfloor cn^2\rfloor -1} \mathbb{P}_1(Z_n = 1|\mathcal{E})}{(1 - \mathbb{P}_1(Z_n = 0|\mathcal{E}))^2} \right] \\ &= n^2 \left[c^2 \frac{\mathbb{P}_1(Z_n = 0|\mathcal{E})^{\lfloor cn^2\rfloor -2} \mathbb{P}_1(Z_n = 1|\mathcal{E})^2}{\mathbb{P}_1(Z_n > 0|\mathcal{E})^3} + c \frac{\mathbb{P}_1(Z_n = 0|\mathcal{E})^{\lfloor cn^2\rfloor -1} \mathbb{P}_1(Z_n = 1|\mathcal{E})}{\mathbb{P}_1(Z_n > 0|\mathcal{E})^2} \right] \\ &= n^2 \left[c^2 \mathbb{P}_1(Z_n = 0|\mathcal{E})^{\lfloor cn^2\rfloor -2} e^{-2S_n} \mathbb{P}_1(Z_n > 0|\mathcal{E}) + c\mathbb{P}_1(Z_n = 0|\mathcal{E})^{\lfloor cn^2\rfloor -1} e^{-S_n} \right].\end{aligned}$$

by using (6.5). Finally, as $\mathbb{P}(Z_n > 0|\mathcal{E}) \leq e^{S_n}$ a.s., we get that a.s.

$$\inf_{z\geq cn^2} \mathbb{P}_z(Z_n = 2|\mathcal{E}) \leq (c^2 + c) n^2 \mathbb{P}_1(Z_n = 0|\mathcal{E})^{\lfloor cn^2\rfloor -1} e^{-S_n}.$$

Next, we use again the change of measure

$$\mathbf{P}(X \in dx) = \frac{e^{-x}\mathbb{P}(X \in dx)}{\mathbb{E}[e^{-X}]}.$$

Then

$$\inf_{z\geq cn^2} \mathbb{P}_z(Z_n = 2) \leq (c^2 + c) n^2 \mathbb{E}[e^{-X}]^n \mathbf{E}[\mathbb{P}(Z_\infty = 0|\mathcal{E})^{\lfloor cn^2\rfloor -2}].$$

Using Jensen's inequality yields

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \inf_{z\geq cn^2} \mathbb{P}_z(Z_n = 2) \leq \log \mathbb{E}[e^{-X}] + \limsup_{n\to\infty} n\mathbf{E}[\log \mathbb{P}(Z_\infty = 0|\mathcal{E})].$$

Finally, note that $\mathbb{E}[Xe^{-X}] > 0$ implies $\mathbf{E}[X] > 0$. Thus $\mathbf{P}(\mathbb{P}(Z_\infty = 0|\mathcal{E}) < 1) > 0$ and therefore $\mathbf{E}[\log \mathbb{P}(Z_\infty = 0|\mathcal{E})] < 0$. This yields the first result.

For the second claim, we use again the geometric form of the probabilities of of LF distributions to get

$$\mathbb{P}_1(1 \leq Z_n \leq cn^2|\mathcal{E}) \leq \lceil cn^2 \rceil \mathbb{P}_1(Z_n = 1|\mathcal{E}).$$

Thus, taking expectations yields

$$\begin{aligned}
\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_1(1 \leq Z_n \leq cn^2) &\leq \limsup_{n\to\infty} \tfrac{1}{n} \log \Big(\lceil cn^2 \rceil \mathbb{P}_1(Z_n = 1) \Big) \\
&= \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_1(Z_n = 1) = \log \mathbb{E}[e^{-X}] \ .
\end{aligned}$$

where the last result has been shown in the previous subsection. □

Lemma 6.7. *We assume that $\mathbb{E}[Xe^{-X}] > 0$. Then, for every $\delta \in (0, 1]$,*

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_1(MRCA_n > \delta n | Z_n = 2) < 0.$$

Proof. First, we recall that the event $\{MRCA_n = \lfloor \delta n \rfloor + 1\}$ implies that there are at least two individuals in generation $n - \lfloor \delta n \rfloor$ and that from this generation on, at least two subtrees survive until generation n. As in the preceding lemmas, we use the branching property and a decomposition according to the two subtrees which survive and get that

$$\begin{aligned}
&\mathbb{P}_1(MRCA_n = \lfloor \delta n \rfloor + 1, Z_n = 2) \\
&\leq \sum_{k=2}^{n^2} \binom{k}{2} \mathbb{P}_1(Z_{n-\lfloor \delta n \rfloor} = k) \mathbb{E}\big[\mathbb{P}_1(Z_{\lfloor \delta n \rfloor} = 1|\mathcal{E})^2 \mathbb{P}(Z_{\lfloor \delta n \rfloor} = 0|\mathcal{E})^{k-2}\big] + \inf_{z\geq n^2} \mathbb{P}_z(Z_{\lfloor \delta n \rfloor} = 2) \\
&\leq A_n + B_n,
\end{aligned}$$

where

$$A_n := n^4 \mathbb{P}_1(1 \leq Z_{n-\lfloor \delta n \rfloor} \leq n^2) \mathbb{E}\big[\mathbb{P}_1(Z_{\lfloor \delta n \rfloor} = 1|\mathcal{E})^2\big], \qquad B_n := \inf_{z\geq n^2} \mathbb{P}_z(Z_{\lfloor \delta n \rfloor} = 2).$$

Letting n go to ∞ and applying Lemma 6.6 yields

$$\begin{aligned}
&\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_1(MRCA_n = \lfloor \delta n \rfloor + 1, Z_n = 2) \\
&\qquad \leq \max\Big\{ \limsup_{n\to\infty} \tfrac{1}{n} \log A_n, \limsup_{n\to\infty} \tfrac{1}{n} \log B_n \Big\} \\
&\qquad = \max\Big\{ \limsup_{n\to\infty} \tfrac{1}{n} \log A_n, -\infty \Big\} = \limsup_{n\to\infty} \tfrac{1}{n} \log A_n. \qquad (6.18)
\end{aligned}$$

Next, let us treat the term named A_n. By Lemma 6.6,

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}(1 \leq Z_{n-\lfloor \delta n \rfloor} \leq n^2) = (1 - \delta) \log \mathbb{E}[e^{-X}]. \qquad (6.19)$$

Using (6.5) and $\mathbb{P}(Z_n > 0|\mathcal{E}) \leq e^{L_n}$ a.s. yields

$$\mathbb{E}[\mathbb{P}_1(Z_{\lfloor \delta n \rfloor} = 1|\mathcal{E})^2] \leq \mathbb{E}[e^{-2S_{\lfloor \delta n \rfloor}} \mathbb{P}_1(Z_{\lfloor \delta n \rfloor} > 0|\mathcal{E})^4] \leq \mathbb{E}[e^{-2S_{\lfloor \delta n \rfloor}+4L_{\lfloor \delta n \rfloor}}]. \qquad (6.20)$$

We recall the change of measure

$$\mathbf{P}(X \in dx) = \frac{e^{-x}\mathbb{P}(X \in dx)}{\mathbb{E}[e^{-X}]}.$$

Then $\mathbb{E}[Xe^{-X}] > 0$ assures that $\mathbf{E}[X] > 0$ and under $\mathbf{P}$, S is a random walk with positive drift. From (6.18), (6.19) and (6.20) we get

$$\begin{aligned}
\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_1(MRCA_n = \lfloor \delta n \rfloor + 1, Z_n = 2)& \\
\leq (1-\delta)\log \mathbb{E}[e^{-X}] &+ \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}[e^{-2S_{\lfloor \delta n \rfloor}+4L_{\lfloor \delta n \rfloor}}] \\
= (1-\delta)\log \mathbb{E}[e^{-X}] &+ \delta \log \mathbb{E}[e^{-X}] + \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbf{E}[e^{-S_{\lfloor \delta n \rfloor}+4L_{\lfloor \delta n \rfloor}}] \\
= \log \mathbb{E}[e^{-X}] &+ \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbf{E}[e^{-S_{\lfloor \delta n \rfloor}+4L_{\lfloor \delta n \rfloor}}].
\end{aligned}$$

As $\mathbb{E}[Xe^{-X}] > 0$, we know from the previous subsection that $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1(Z_n = 2) = \log \mathbb{E}[e^{-X}]$. So the previous inequality yields

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_1(MRCA_n = \lfloor \delta n \rfloor + 1 | Z_n = 2) \leq \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbf{E}[e^{-S_{\lfloor \delta n \rfloor}+4L_{\lfloor \delta n \rfloor}}].$$

Finally, we prove that $\limsup_{n\to\infty} \frac{1}{n} \log \mathbf{E}[e^{-S_{\lfloor \delta n \rfloor}+4L_{\lfloor \delta n \rfloor}}] < 0$ to conclude. Decomposing the expectation with $0 < c < \mathbf{E}[X]$ and using $4L_{\lfloor \delta n \rfloor} - S_{\lfloor \delta n \rfloor} \leq 0$ yields

$$\begin{aligned}
\mathbf{E}[e^{-S_{\lfloor \delta n \rfloor}+4L_{\lfloor \delta n \rfloor}}] &\leq e^{-c\lfloor \delta n \rfloor} + \mathbf{P}(4L_{\lfloor \delta n \rfloor} - S_{\lfloor \delta n \rfloor} > -c\lfloor \delta n \rfloor) \\
&\leq e^{-c\lfloor \delta n \rfloor} + \mathbf{P}(S_{\lfloor \delta n \rfloor} < c\lfloor \delta n \rfloor).
\end{aligned}$$

As $0 < c < \mathbf{E}[X]$, by standard results of large deviation theory, the probability on the right-hand side is exponentially small if $\mathbf{E}[e^{-sX}] = \mathbb{E}[e^{(-1-s)X}] < \infty$ for some $s > 0$. This yields that

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_1(MRCA_n = \lfloor \delta n \rfloor + 1 | Z_n = 2) < 0$$

and thus

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_1(MRCA_n > \delta n | Z_n = 2) < 0.$$

□

7 Examples with two environments : dependence on the initial and final population.

In this section, we focus on the importance of the initial population.

Example 1 : the limits of $\frac{1}{n} \log \mathbb{P}_1(Z_n = i)$ and $\frac{1}{n} \log \mathbb{P}_1(Z_n = j)$ may be both finite, negative but different. Assume that the environment consists of two states q_1 and q_2 such that

$$r := \mathbb{P}(Q_1 = q_1) = 1 - \mathbb{P}(Q_1 = q_2) > 0; \quad q_1(1) = 1; \quad q_2(0) = p,\ q_2(2) = 1-p,$$

with $p \in (0,1)$. Then

$$\tfrac{1}{n} \log \mathbb{P}_1(Z_n = 1) = \log r, \qquad \tfrac{1}{n} \log \mathbb{P}_1(Z_n = 2) \geq \max\big\{ \log r; \log[(1-r)2(1-p)p] \big\}.$$

where the term $\log r$ comes from a population which stays equal to 1 in the environment sequence $(q_1, q_1, q_1, \cdots)$ and the last term comes from a population which stays equal to 2 in the environment sequence $(q_2, q_2, q_2, \cdots)$. Thus if r is chosen small enough (i.e. $r < \frac{2(1-p)p}{1+2(1-p)p}$),

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1(Z_n = 1) < \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1(Z_n = 2).$$

Example 2 : the limits of $\frac{1}{n}\log \mathbb{P}_k(Z_n = k)$ and $\frac{1}{n}\log \mathbb{P}_1(Z_n = k)$ with $k > 1$ may be both finite, negative but different. Actually, in the case without extinction, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_i(Z_n = k) = i \log \mathbb{P}(Z_1 = 1),$$

as soon as $\mathbb{P}_i(\exists n : Z_n = k) > 0$ and the result is immediate. We give here an example with $k = 2$ and possible extinction. We first observe that such an example is not possible with one environment, i.e. in the Galton-Watson case. Then we introduce a simple example in the random environment case and check that it is not in contradiction to Theorem 2.1, before considering the asymptotic behavior of the probabilities involved.

Indeed, for Galton-Watson processes with reproduction law q such that $q(0) > 0$ and $q(2) > 0$, the fact that $q(1) > 0$ already ensures that the limits of $\frac{1}{n}\log \mathbb{P}_2(Z_n = 2)$ and $\frac{1}{n}\log \mathbb{P}_1(Z_n = 2)$ are equal. In the case $q(1) = 0$, we get that

$$\mathbb{P}_1(Z_{n+1} = 2) = \mathbb{P}_1(Z_1 \geq 2, Z_{n+1} = 2) = \sum_{i\geq 2} q(i)\mathbb{P}_i(Z_n = 2)$$

whereas killing one of the initial individuals starting from 2 and letting the other survive yields :

$$\mathbb{P}_2(Z_{n+1} = 2) \geq 2q(0) \sum_{i\geq 2} q(i)\mathbb{P}_i(Z_n = 2).$$

Thus $\mathbb{P}_2(Z_{n+1} = 2) \geq 2q(0)\mathbb{P}_1(Z_n = 2)$. A converse inequality is clear, so the limits of $\frac{1}{n}\log \mathbb{P}_2(Z_n = 2)$ and $\frac{1}{n}\log \mathbb{P}_1(Z_n = 2)$ have to be equal in the Galton-Watson case with possible extinction.
Thus, we consider two environments to provide an example that the initial population size is also of importance even if extinction is possible. More precisely, let the environment consist of the two states q_1 and q_2 such that

$$\begin{aligned}
&r := \mathbb{P}(Q_1 = q_1) = 1 - \mathbb{P}(Q_1 = q_2) > 0,\\
&q_1(1) = p \quad , \quad q_1(a) = 1 - p,\\
&q_2(0) = p \quad , \quad q_2(2) = p \quad , \quad q_2(a) = 1 - 2p,
\end{aligned}$$

with $p \in (0, \frac{1}{2})$ and $a > 2$. Note $k = 1 \notin Cl(\mathcal{I})$, so this example doesn't contradict Theorem 2.1 where it is assumed that the initial population size is in $Cl(\mathcal{I})$.
To prove that $\mathbb{P}_1(Z_n = 2) \gg \mathbb{P}_2(Z_n = 2)$, we first observe that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1(Z_n = 2) \geq \log(rp), \tag{7.1}$$

which comes from a population staying equal to 1 before the last generation in the environment sequence $(q_1, q_1, \ldots, q_1, q_2)$.

Next, let us estimate the extinction probability, given the environment. We first observe that any

BPVE whose environments are either q_1 or q_2 is stochastically larger than the Galton-Watson process with reproduction law (and unique environment) q_2. As a consequence,

$$\mathbb{P}_1(Z_n = 0|\mathcal{E}) \leq \mathbb{P}_1(Z_n = 0|Q_1 = q_2 \dots, Q_n = q_2) \leq \mathbb{P}(Z_\infty = 0|Q_1 = q_2, Q_2 = q_2, \dots) =: s_e \quad \text{a.s.}$$

It is well-known that s_e is given as the first fix point of the generating function f_2 of q_2:

$$s_e = f_2(s_e) = p + ps_e^2 + (1-2p)s_e^a.$$

Let us now estimate s_e. For $s = 2p$, we have $2p > f_2(2p) = p + 4p^3 + (1-2p)2^a p^a$ if a is large enough since $p < 1/2$. Thus $s_e \leq 2p$ if only a is large enough.
We get then for all $i \geq 1$, $k \leq n$,

$$\mathbb{P}(Z_n = 0|\mathcal{E}, Z_k = i) \leq s_e^i \leq (2p)^i \quad \text{a.s.}$$

Using this estimate and the explicit law of $\mathbb{P}(Z_{k+1} = \cdot \mid Z_k = 2, Q_k = q_1)$, we obtain a.s.

$$\begin{aligned}
&\mathbb{P}_2(Z_n = 2|\mathcal{E}, Q_k = q_1, Z_k = 2) \\
&\quad = p^2\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2) + 2(1-p)p\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 1+a) \\
&\qquad\qquad + (1-p)^2\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2a) \\
&\quad = \mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2)\Big(p^2 + 2(1-p)p\binom{a+1}{2}\mathbb{P}(Z_n = 0|\mathcal{E}, Z_{k+1} = 1+a-2) \\
&\qquad + (1-p)^2\binom{2a}{2}\mathbb{P}(Z_n = 0|\mathcal{E}, Z_{k+1} = 2a-2)\Big) \\
&\quad \leq \mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2)\Big(p^2 + 2(1-p)p\binom{a+1}{2}(2p)^{a-1} + (1-p)^2\binom{2a}{2}(2p)^{2a-2}\Big)
\end{aligned}$$

If p is small enough (depending on $a > 2$), we get that a.s.

$$\mathbb{P}_2(Z_n = 2|\mathcal{E}, Q_k = q_1, Z_k = 2) \leq \mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2)3p^2.$$

Analogously, if the environment q_2 occurs in generation k, we get that a.s.

$$\begin{aligned}
&\mathbb{P}_2(Z_n = 2|\mathcal{E}, Q_k = q_2, Z_k = 2) \\
&\quad = 2p^2\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2) + p^2\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 4) + 2p(1-2p)\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = a) \\
&\qquad + 2p(1-p)\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = a+2) + (1-2p)^2\mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2a) \\
&\quad \leq \mathbb{P}(Z_n = 2|\mathcal{E}, Z_{k+1} = 2)3p^2.
\end{aligned}$$

Next, note that the population starting from $Z_0 = 2$ is either always ≥ 2 or extinct. Thus in each generation, there are at least two individuals and we may apply the estimates above for the subtrees emerging in generation k. Finally we get that

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_2(Z_n = 2) = \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{E}\big[\mathbb{P}_2(Z_n = 2|\mathcal{E})\big] \leq \log(3p^2).$$

We now choose p small enough such that $3p^2 < rp$ and recall (7.1) to get

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_2(Z_n = 2) < \liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}_1(Z_n = 2).$$

Finally, we note that this example shows that, as in the the case without extinction in [9], the initial population may be of importance for the asymptotic of the probability of staying small, but

alive.

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